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EFFICIENT MULTINOMIAL SELECTION IN SIMULATION

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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$* * * * *$

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ABSTRACT

Consider a simulation experiment consisting of v independent vector observations or replications across k systems, where in any given replication one and only one system is selected as the best performer (i.e., it wins) based on some performance measure. Each system has an unknown constant probability of winning in any replication and the numbers of wins for the individual systems follow a multinomial distribution. The classical multinomial selection procedure of Bechhofer, Elmaghraby, and Morse (Procedure BEM), prescribes a minimum number of replications, denoted as v^*, so that the probability of correctly selecting the true best system meets or exceeds a prespecified probability. Assuming that larger is better, Procedure BEM selects as best the system having the largest value of the performance measure in more replications than any other system.

In this research, we use these same v^* replications across k systems to form $(v^*)^k$ pseudo-replications that contain one observation from each system, and develop Procedure AVC (All Vector Comparisons) to achieve a higher probability of correct selection (PCS) than with Procedure BEM. For specific small-sample cases and via a large-sample approximation we show that the PCS with Procedure AVC exceeds

ii

the PCS with Procedure BEM. In a similar fashion, we show that with Procedure AVC we achieve a given PCS with a smaller v than the v^* required with Procedure BEM.

We also consider the closely related problem of estimating how likely each system is to be the best under both procedures. Surprisingly, estimating $p_{[k]}$ is a different problem than estimating p_j, $j = 1, \ldots, k$. We show that the variance of the AVC estimator is never larger than the variance of the BEM estimator (the standard Maximum Likelihood Estimator) and quantify the reduction in variance with the AVC estimator for specific small-sample cases and asymptotically.

To my Mom and Dad — Thanks for your faith in me and constant

encouragement.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

We consider the problem of selecting the best of a set of alternatives or systems based upon a comparison among them. To begin, we consider an experiment consisting of a series of independent trials across all systems where, on any given trial, one and only one system is selected as the best performer (i.e., it wins). Each system has an unknown constant probability of winning on any trial. We count the number of wins for each system over all the trials and select the system with the most wins as the best system. In such an experiment, the numbers of wins for the individual systems follow a multinomial distribution. The problem of determining which of the systems has the largest probability of being the best is known as the multinomial selection problem (MSP).

In our context, we require a quantitative measure of the performance of each system on each trial. Specifically, we consider the problem of determining which of k simulated systems is most likely to be the best performer. A standard experiment is to generate v independent vector observations or replications (i.e., trials) across the k systems. Each vector replication produces a single performance measure for each system.

Consider the following example. Suppose we are tactical war planning analysts who are directed to provide the Joint Task Force Commander with the best plan to cripple the enemy's command and control. "Best" means achieving the highest level of cumulative damage expectancy (CDE) against a selected set of targets given current intelligence estimates of enemy defense capabilities and available friendly forces. Our team prepares four distinct attack plans and we simulate v independent replications across all four plans. For each replication we compare the CDE among each of the four plans. Since the chosen plan can only be executed a single time, we select as the best plan the one that has the largest CDE in the most replications.

Our goal in an MSP is to find the system that is most likely to be the best performer among the systems, as opposed to identifying the best average performer in the long run, with a minimum amount of data. A classical solution procedure for the MSP, Procedure BEM (Bechhofer, Elmaghraby, and Morse 1959), prescribes a minimum number of independent vector replications, v^*, across all systems so that the probability of correctly selecting the true best system meets or exceeds a prespecified probability. Assuming that larger is better, BEM selects as best the system having the largest value of the performance measure in more replications than any other.

We propose a new solution procedure for the MSP that requires no additional data, but is designed to increase the probability that the best system is so identified. Specifically, we propose using the same v^* replications across k systems for BEM to form $(v^*)^k$ pseudo-replications, each containing one observation from each

2

system, and selecting as best the system having the largest value of the performance measure in more pseudo-replications than any other system. For specific small-sample cases and via a large-sample approximation, we show that this new procedure, Procedure AVC (All Vector Comparisons), dominates BEM in the sense that AVC never requires more independent replications than BEM to meet a pre-specified probability of correct selection (PCS). AVC represents a more efficient use of the available data. From a simulation design point of view, we also show that by using AVC we can achieve a given PCS with fewer replications than are required to reach the same PCS with BEM.

MSP applications where AVC can be applied include selecting the best of a set of tactical or strategic military actions as presented earlier. An example in the area of structural engineering is finding the design that performs best in a one-time catastrophic event, such as an earthquake. Some specific simulation examples include selecting the schedule most likely to result in completing all jobs on time; selecting the investment portfolio most likely to provide the largest return; or selecting the computer system with the highest probability of completion of a series of tasks without failure. Each of these applications involves the comparison of quantitative measures of performance among competing systems, as opposed to comparing qualitative differences. AVC requires a quantitative measure of system performance for each system in each trial to be compared with the performance measure of other systems across any or all of the remaining trials.

In some MSPs, the performance of the alternatives is measured qualitatively, rather than quantitatively. Thus, rather than a comparison based on a numerical measure of performance, we may only know which alternative won on any given

3

trial. For example, in the areas of marketing research or opinion surveys, we might be interested in determining the most popular brand, flavor, etc., or the most favored candidate or position on a political issue. In such examples, we cannot compare the preference for an individual alternative in one trial with the preference for one of the remaining alternatives in another trial. For Procedure AVC we only consider quantitative comparisons.

In an MSP we measure success in terms of the probability we select the true best system, our PCS. However, the PCS contains no information about how likely the system we selected as best is to actually be the best. In order to obtain this additional information, we can obtain a point estimate for the probability of a particular system winning on any given replication. These individual probabilities are multinomial success probabilities, and we obtain different estimators for these probabilities using BEM and AVC. The expected values of these estimators are the same for both methods; however, we show that we obtain a variance reduction with AVC over BEM and quantify this reduction for specific cases. In this light, we also compare AVC and BEM in terms of the precision of the point estimators provided by each method.

This paper is organized into two chapters, each written in the form of a self-contained journal article. Chapter 2 covers the multinomial selection problem and our new approach to solving it, Procedure AVC. Chapter 3 considers the related point estimation problem and extends our results into this context. Appendix A contains proofs, Appendix B presents additional simulation results, and Appendix C provides the source code used.

CHAPTER 2

EFFICIENT MULTINOMIAL SELECTION IN SIMULATION

2.1 Introduction

Suppose we have $k \geq 2$ independent populations, denoted $\pi_1, \pi_2, \ldots, \pi_k$. In a simulation context each population is a simulated system. We consider the problem of selecting the best of the k systems based on simulated results for all of the systems.

Let X_{ji} represent the ith replication from system j of some performance measure. Each system (π_j, $j = 1, \ldots, k$) has an unknown constant probability (p_j, $j = 1, \ldots, k$) of having the largest value of the performance measure. Assume we have generated v independent replications from each of the k systems. We define the best system as the system most likely to have the largest performance measure in any comparison across all systems. Such a comparison corresponds to a multinomial trial, where one and only one system can win in any given trial. Our objective is to find the system that is most likely to be the best performer in a single trial

among the systems, as opposed to identifying the best average performer over the long run, with a minimum amount of data. This is known as the multinomial selection problem (MSP).

A classical solution procedure for the MSP, Procedure BEM (Bechhofer, Elmaghraby, and Morse 1959), prescribes a minimum number, v^*, of independent vector replications across all systems so that the probability of correctly selecting the true best system meets or exceeds a prespecified probability. Assuming that larger is better, BEM selects as best the system having the largest value of the performance measure in more replications than any other.

MSP applications include selecting the best of a set of tactical or strategic military actions. An example in the area of structural engineering is finding the design that performs best in a one-time catastrophic event, such as an earthquake. Simulation examples include selecting the schedule most likely to result in completing all jobs on time; selecting the investment portfolio most likely to provide the largest return; or selecting the computer system with the highest probability of completing a series of tasks without failure. Each of these applications involves the comparison of quantitative measures of performance among competing systems as opposed to comparing qualitative measures. For the type of MSP considered in this study, we require a quantitative measure of system performance for each system in each trial to be compared with the performance of other systems across any or all of the remaining trials.

Let $\mathbf{X_i} = (X_{1i}, X_{2i}, \ldots, X_{ki})$ represent the ith replication across all k systems. Let $Y_{ji} = 1$ if $X_{ji} > X_{\ell i}$, for $\ell = 1, 2, \ldots, k$, but $\ell \neq j$; and let $Y_{ji} = 0$ otherwise. In other words, $Y_{ji} = 1$ if X_{ji} is the largest observation in $\mathbf{X}_i$. In case of a

tie for the largest value, we randomly select one of the tied populations as the best. In terms of our simulation example involving the best investment portfolio, X_{1i} represents the return generated for Portfolio 1 on the ith replication. If the return on Portfolio 1 exceeds the returns of all the remaining portfolios in that replication, then Portifolio 1 wins ($Y_{1i} = 1$) and all the remaining portfolios lose ($Y_{ji} = 0;\ j = 2, 3, \ldots, k$). If more than one portfolio ties for the largest return then we randomly select one of these to be the winner in that replication.

Suppose that there are v independent replications across all systems, and let $Y_j = \sum_{i=1}^{v} Y_{ji}$ represent the number of times system j wins out of these v replications. Let $p_j = \Pr\{X_{ji} > X_{\ell i}, \forall \ell \neq j\}$ where $0 < p_j < 1$ and $\sum_{j=1}^{k} p_j = 1$. Then $\sum_{j=1}^{k} Y_j = v$ and the k-variate discrete random variable $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)$ follows a multinomial distribution with success probabilities $\mathbf{p} = (p_1, p_2, \ldots, p_k)$. Therefore, the probability mass function for $\mathbf{Y}$ with parameters v and $\mathbf{p}$ is

$$\Pr\{Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k) = \frac{v!}{\prod_{j=1}^{k} y_j!} \prod_{j=1}^{k} p_j^{y_j}.$$

For a standard MSP with v replications from each system, we have v independent comparisons (trials) to use in selecting the best system. For an experiment involving physical measurements of system performance at common times, it makes sense to group the observations across systems for the same trial due to possible variations in extraneous factors beyond the experimentor's control. Also, in some instances, the performance of the competing systems may be measured qualitatively, or may only indicate which system won in a given trial. In such cases, there is no quantitative measure we can compare across systems in different replications. Examples include marketing research or opinion surveys where the data collected are categorical (e.g., favorite brand, food, or political candidate).

7

Due to convention and convenience when comparing simulated system responses, the responses are typically grouped by replication, corresponding to a trial in a physical experiment. Grouping system responses in this fashion is arbitrary and since our simulated responses are quantitative, we can compare any observation from one system with any observation from each of the remaining systems. This means that a single observation from system 1 can be grouped in a vector comparison with any one of the v observations from system 2, and with any one of the v observations from system 3, and so on, up to and including any one of the v observations from system k. Since there are v observations from system 1 as well, this gives us a total of v^k vector comparisons (trials) that can be formed with v independent observations from the k systems. We incorporate this setup in a new MSP procedure, which we call AVC, for $\underline{A}$ll $\underline{V}$ector $\underline{C}$omparisons. By performing only the v vector comparisons where the observations for each system are from the same replication, as is done with BEM, we disregard the information available from the remaining $v^k - v$ comparisons.

Our results suggest a number of advantages of AVC over BEM. For specific small-sample examples, we show that AVC has a larger probability of correct selection (PCS) than BEM for a fixed v. We show this analytically for small values of v and k, and also present simulation results for up to $k = 10$ systems and $v = 50$ vector replications. Looking at these results from a slightly different perspective, we also demonstrate acheivement of a desired PCS with a smaller value of v when

8

using AVC as compared to BEM. The first perspective emphasizes a more efficient use of the available data to increase PCS. The second view points towards a more efficient way to design a simulation experiment using the smallest value of v required to achieve a desired PCS.

Unlike BEM, the PCS for AVC depends on the distributions of the simulation outputs, not just on $p_1, \ldots, p_k$. However, we also show that the dependence is weak. This fact, along with the difficulty of analytically evaluating the PCS of AVC for even small k and v, leads us to a large-sample approximation (LSA) for the PCS using AVC. As $v \to \infty$, any distributional differences in PCS with AVC disappear. Therefore, our LSA is distribution independent and converts an AVC problem into an equivalent BEM problem. Our LSA demonstrates that asymptotically the PCS with AVC is larger than the PCS with BEM. Additionally, this LSA shows that AVC can provide better discrimination between the systems at the same level of confidence and with the same data.

This paper is organized as follows: We first provide a brief review of the MSP and the classical approach to solving it. Then we describe our new procedure, AVC, and present analytical results covering a variety of specific population distributions for the performance measures. Our LSA is then presented by recasting PCS in terms of a point estimation problem for the multinomial success probabilities, $p_j, \ j = 1, \ldots, k$. Empirical results follow for specific distributions and include simulations designed to test the robustness of our LSA.

2.2 Background

Bechhofer, Elmaghraby and Morse (1959) describe a single-stage procedure for selecting the multinomial event (population or system) which has the largest success probability. BEM requires the specification of P^* (where $1/k < P^* < 1$), a minimum probability of correctly identifying the population with the largest success probability (i.e., the best population), and θ^* (where $1 < \theta^* < \infty$), the minimum ratio of the largest success probability to the second largest success probability that we want to be able to detect. The procedure, as adapted to simulation, consists of the following steps:

Procedure 2.1 *(BEM)*

1. *For given k and θ^*, find the minimum value of v, denoted v^*, that guarantees that the PCS is at least P^*.*

2. *Generate v^* independent replications for each population.*

3. *Compute $Y_j = \sum_{i=1}^{v^*} Y_{ji}$, for $j = 1, 2, \ldots, k$.*

4. *Let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(k)}$ be the ranked sample counts from step 3. Select the population associated with the largest count, $Y_{(k)}$, as the best population. In case of a tie for the largest count, randomly select one of the tied populations as the best.*

To determine the appropriate v^* in step 1, let $p_{[1]} \leq p_{[2]} \leq \cdots \leq p_{[k]}$ denote the ranked success probabilities for the k populations. Since only values of the ratio

$\theta = p_{[k]}/p_{[k-1]}$ greater than or equal to θ^* are of interest, we are indifferent between the best and the next-best population for values of $\theta < \theta^*$. A procedure of this type is referred to as an *indifference-zone approach*. Select v^* as the minimum number of independent vector observations required to achieve a PCS greater than or equal to P^* whenever $\theta \geq \theta^*$.

We define the least favorable configuration (LFC) of $\mathbf{p} = (p_{[1]}, p_{[2]}, \ldots, p_{[k]})$ as the configuration where PCS is a minimum over all configurations with $\theta \geq \theta^*$ (Gibbons, Olkin, and Sobel 1977). If we obtain a PCS $\geq P^*$ with our selected v^* under the LFC, then a PCS of at least P^* can be guaranteed for *any* configuration of $\mathbf{p}$ with $\theta \geq \theta^*$. Keston and Morse (1959) prove that the LFC for BEM is given by

$$p_{[1]} = p_{[2]} = \cdots = p_{[k-1]} = \frac{1}{\theta^* + k - 1}$$
$$p_{[k]} = \frac{\theta^*}{\theta^* + k - 1}. \tag{2.1}$$

Although we only need to consider the LFC for designing sampling plans, the PCS can be calculated for any $\mathbf{p}$ with $p_{[k]} > p_{[k-1]}$ as follows.

Let $\pi_{[j]}$ be the population associated with $p_{[j]}$ and let $y_{[j]}$ represent the number of wins for $\pi_{[j]}$. Thus, the subscripts for the populations and the associated number of wins are based on the ranking of the p_js. We refer to the PCS using BEM for a fixed k and v as PCS^{bem}. For any fixed k and v, PCS^{bem} can be expressed as

$$\text{PCS}^{\text{bem}}(\mathbf{p}) = \sum_{\mathbf{y}} \frac{1}{t(\mathbf{y})} \frac{v!}{\prod_{j=1}^{k} y_{[j]}!} \prod_{j=1}^{k} p_{[j]}^{y_{[j]}},$$

where the summation is over all vectors $\mathbf{y} = (y_{[1]}, \ldots, y_{[k]})$ such that $\sum_{j=1}^{k} y_j =$

v, $y_{[k]} \geq y_{[j]}$ $(j = 1, 2, \ldots, k-1)$, where $t(\mathbf{y})$ is a function of $y_{[1]}, \ldots, y_{[k]}$, representing the number of populations tied for the most wins (Bechhofer, Elmaghraby, and Morse 1959).

2.3 All Vector Comparisons (AVC)

We propose a method to provide a PCS greater than or equal to PCS^{bem} (in at least some cases) using the same replications $\mathbf{X}_i, i = 1, 2, \ldots, v$. We use the BEM parameters k, P^*, and θ^*, and we execute the first step of BEM to find a value of v^*. However, rather than comparing the ith replication for each system with the ith replications of the other systems, consider instead a total of $(v^*)^k$ pseudo-replications formed by associating each X_{ji} $(j = 1, 2, \ldots, k; \; i = 1, 2, \ldots, v^*)$, with all possible combinations of the remaining $X_{\ell h}$ $(\ell = 1, 2, \ldots, k; \; \ell \neq j; \; h = 1, 2, \ldots, v^*)$. Each such pseudo-replication contains one observation from each population. Note that the $(v^*)^k$ pseudo-replications include the v^* independent replications from which the pseudo-replications are formed.

Define
$$Z_j = \sum_{a_1=1}^{v} \sum_{a_2=1}^{v} \cdots \sum_{a_k=1}^{v} \prod_{\ell=1; \ell \neq j}^{k} \phi(X_{ja_j} - X_{\ell a_\ell}) \qquad (2.2)$$
for $j = 1, 2, \ldots, k$ with

$$\phi(a) = \begin{cases} 1, & a > 0 \\ 0, & a < 0 \\ \text{randomly assign} \\ 0 \text{ or } 1, & a = 0. \end{cases}$$

Thus, Z_j represents the number of times out of v^k pseudo-replications that population π_j wins (ties broken randomly) and $\sum_{j=1}^{k} Z_j = v^k$.

As a specific illustration of how the pseudo-replications are formed, consider $k = 3$ systems with $v = 2$ observations generated for each. Our original replications and counts $(Y_j, j = 1, 2, 3)$ are then

$$
\begin{array}{ccc}
X_{11} & X_{21} & X_{31} \\
X_{12} & X_{22} & X_{32} \\
\hline
Y_1 & Y_2 & Y_3
\end{array}
$$

We will have a total of v^k ($2^3 = 8$) pseudo-replications including our two original replications when using AVC. These pseudo-replications and the associated counts $(Z_j, j = 1, 2, 3)$ are

$$
\begin{array}{ccc}
X_{11} & X_{21} & X_{31} \\
X_{11} & X_{21} & X_{32} \\
X_{11} & X_{22} & X_{31} \\
X_{11} & X_{22} & X_{32} \\
X_{12} & X_{21} & X_{31} \\
X_{12} & X_{21} & X_{32} \\
X_{12} & X_{22} & X_{31} \\
X_{12} & X_{22} & X_{32} \\
\hline
Z_1 & Z_2 & Z_3
\end{array}
$$

Our new procedure consists of the following steps:

Procedure 2.2 *(AVC)*

1. *Given values for k, P^*, and θ^*, use step 1 of Procedure BEM to determine a value for v^*.*

2. *Generate v^* independent replications for each population and construct the additional $(v^*)^k - v^*$ pseudo-replications possible with one value from each of the populations.*

3. *Compute Z_j using Equation (2.2).*

13

4. Let $Z_{(1)} \leq Z_{(2)} \leq \cdots \leq Z_{(k)}$ be the ranked sample counts from step 3. Select the population associated with the largest count, $Z_{(k)}$, as the best population. In case of a tie for the largest count, randomly select one of the tied populations as the best.

Later we demonstrate for specific cases that the PCS with AVC, referred to as PCS^{avc}, is greater than or equal to PCS^{bem}.

As written, step 1 of Procedure AVC uses the same number of replications as BEM. Suppose we modify step 1 to use the minimum v where $\text{PCS}^{\text{avc}} \geq P^*$. We demonstrate later that a smaller number of replications are required with AVC relative to BEM to achieve P^*. We provide such values of v in this paper.

PCS^{avc} can be expressed as

$$\text{PCS}^{\text{avc}}(\mathbf{p}) = \sum_{(\mathbf{z})} \frac{1}{t(\mathbf{z})} \Pr\{Z_{[1]} = z_{[1]}, \ldots, Z_{[k]} = z_{[k]}\},$$

where the summation is over all vectors $\mathbf{z} = (z_{[1]}, \ldots, z_{[k]})$ such that $\sum_{j=1}^{k} z_j = v^k$, $z_{[k]} \geq z_{[j]}$, $j = 1, 2, \ldots, k-1$, where $t(\mathbf{z})$ is a function of $z_{[1]}, \ldots, z_{[k]}$, representing the number of populations tied for the most wins. Unfortunately, $\mathbf{Z}$ does not follow a multinomial distribution, so we must refer to the distributions of the original observations, X_{ji}, to calculate PCS^{avc}. Analytical and simulation results using a number of different population distributions show that PCS^{avc} depends weakly on the underlying distributions of the X_{ji}s.

2.4   Analytical Results

The following analytical study illustrates a number of important properties of the AVC method. First, we demonstrate the improvement possible with AVC for specific cases. We also show a weak dependence in the AVC results on the underlying population distributions for the X_{ji}. Lastly, we demonstrate the difficulty in obtaining analytical results for even a small number of populations and observations, and thus provide motivation for our large sample approximation of PCS^{avc} which is distribution independent.

2.4.1   Small-Sample Results

Initially, we restrict our attention to continuous distributions for the $X_{ji}s$, which eliminates the possibility of ties among the observations. We let $\pi_{[k]}$ be the best population and assume all the remaining populations, $\pi_{[1]}, \ldots, \pi_{[k-1]}$, are identically distributed. This setup gives us the LFC for BEM. We also consider all population distributions to belong to the same parametric family. We will calculate PCS^{avc} by conditioning on the joint density of all the order statistics for the v independent replications from $\pi_{[k]}$.

Consider a set of v vector replications across all populations. Combine all the observations from all populations and rank them from smallest to largest. Refer to each observation by its rank and consider permutations of these ranks. For any such permutation we can determine the value of $Z_{[k]}$ and calculate the probability

15

of obtaining that arrangement of ranks. We refer to such an arrangement as a *rank order*. Recall that $Z_{[k]}$ represents the number of times the best population, $\pi_{[k]}$, wins out of the v^k pseudo-replications. For illustrative purposes, let X represent an observation from $\pi_{[k]}$ and let O represent an observation from any of the remaining inferior populations.

As an example, suppose $k = 3, v = 2$. Then

$$\Pr\{Z_{[3]} = 8\} = \Pr\{O_{(1)} < O_{(2)} < O_{(3)} < O_{(4)} < X_{(1)} < X_{(2)}\} \qquad (2.3)$$

$$\Pr\{Z_{[3]} = 6\} = 4\,\Pr\{O_{(1)} < O_{(2)} < O_{(3)} < X_{(1)} < O_{(4)} < X_{(2)}\} \qquad (2.4)$$

Since we do not know which of the four Os is associated with which rank, we must account for all permutations of the Os that result in a different combination of adjacent Os. For probability statement (2.3), there is only one combination of adjacent Os from the 4! permutations of the Os that is less than both Xs. In the rank order for probability statement (2.4), since any one of the Os can be associated with $O_{(4)}$, we have four distinct combinations (in terms of which set of Os are adjacent) that result in this one rank order. This is why the coefficient '4' appears on the right-hand side of Equation (2.4). In general this coefficient is $\binom{n}{r}$, where $n = v(k - 1)$ is the total number of observations from the inferior populations and r is the largest number of these observations that are adjacent. Similar arguments can be used to derive expressions for possible values of $Z_{[k]}$ for integers $k, v \geq 2$. For this example, there is only a single rank order that results in each value of $Z_{[k]}$. As k or v get even moderately large, there will be multiple rank orders that result in the same value for $Z_{[k]}$. In addition, for larger k and v we must also take into consideration how many of the combinations contain an

16

observation from each of the inferior populations. Therefore, the calculation of the probability of each value of $Z_{[k]}$ becomes extremely tedious with increasing k or v.

Restricting our attention to $k = 2$ populations, it is interesting to note that the vector comparisons with AVC are analagous to the comparisons that form the Wilcoxon rank-sum statistic (Randles and Wolfe 1979). Let W equal the sum of the ranks of the observations from the best population. Then W is the Wilcoxon rank-sum statistic and our $Z_{[2]}$ is the Mann-Whitney U statistic. Therefore, W can be expressed as a function of our $Z_{[2]}$ as

$$W = Z_{[2]} + \frac{v}{2}(v + 1).$$

In terms of W, AVC always makes a correct selection for $W > \mathrm{E}[W]$ (incorrect selection for $W < \mathrm{E}[W]$), where $\mathrm{E}[W]$ is the expected value of W under the assumption that the two populations are identical in distribution.

If we specify a particular distribution family for our populations, then we can derive formulas to compare $\mathrm{PCS}^{\mathrm{avc}}$ with $\mathrm{PCS}^{\mathrm{bem}}$ for small k and v. We present results for exponential, continuous uniform, and Bernoulli distributions.

2.4.2 Exponential

First, consider $X \sim \exp(\lambda)$ and $O \sim \exp(\mu)$ and let $\lambda < \mu$, where $\lambda > 0$ and $\mu > 0$ are exponential rates. This particular example was suggested by Goldsman (1995) in some early discussions concerning this research. For $k = 2$, $v = 2$, we have $p_{[2]} = \Pr\{X > O\} = \mu/(\lambda + \mu)$ and $p_{[1]} = \Pr\{X < O\} = \lambda/(\lambda + \mu)$. To calculate $\mathrm{PCS}^{\mathrm{bem}}$, we need to consider vectors $\mathbf{Y} = (Y_{[1]}, Y_{[2]})$ such that $Y_{[2]} \geq Y_{[1]}$. With $v = 2$, the only possible $\mathbf{Y}$s with $Y_{[2]} \geq Y_{[1]}$ are $(0, 2)$ and $(1, 1)$. This gives us

$$
\begin{aligned}
\mathrm{PCS}^{\mathrm{bem}} &= \Pr\{Y_{[2]} = 2\} + \frac{1}{2}\Pr\{Y_{[2]} = 1\} \\
&= p_{[2]}^2 + \frac{1}{2}2\,p_{[1]}p_{[2]} \\
&= \frac{\mu}{\lambda + \mu}.
\end{aligned}
\tag{2.5}
$$

Similarly, to calculate $\mathrm{PCS}^{\mathrm{avc}}$, we need to consider vectors $\mathbf{Z} = (Z_{[1]}, Z_{[2]})$ such that $Z_{[2]} \geq Z_{[1]}$. With $v^k = 4$, the only $\mathbf{Z}$s with $Z_{[2]} \geq Z_{[1]}$ are: $(0, 4)$, $(1, 3)$, $(2, 2)$. So

$$
\mathrm{PCS}^{\mathrm{avc}} = \Pr\{Z_{[2]} = 4\} + \Pr\{Z_{[2]} = 3\} + \frac{1}{2}\Pr\{Z_{[2]} = 2\}.
\tag{2.6}
$$

When $X \sim \exp(\lambda)$, the joint distribution of $(X_{[1]}, X_{[2]})$ is $f_{12}(a, b) = 2\lambda^2 e^{-\lambda(a+b)}$. The probabilities on the right-hand side of Equation (2.6) can then be found as follows:

$$\Pr\{Z_{[2]} = 4\} = \Pr\{O_{(1)} < O_{(2)} < X_{(1)} < X_{(2)}\}$$

$$= \int_0^\infty \int_0^b (1 - e^{-\mu a})^2 2\lambda^2 e^{-\lambda(a+b)} \, da \, db$$

$$= \frac{\mu^2}{(2\lambda + \mu)(\lambda + \mu)}$$

$$\Pr\{Z_{[2]} = 3\} = 2 \int_0^\infty \int_0^b (1 - e^{-\mu a})(e^{-\mu a} - e^{-\mu b}) 2\lambda^2 e^{-\lambda(a+b)} \, da \, db$$

$$= \frac{2\lambda\mu^2}{(2\lambda + \mu)(\lambda + \mu)^2}$$

$$\Pr\{Z_{[2]} = 2\} = \int_0^\infty \int_0^b (e^{-\mu a} - e^{-\mu b})^2 2\lambda^2 e^{-\lambda(a+b)} \, da \, db \; +$$

$$2 \int_0^\infty \int_0^b (1 - e^{-\mu a}) e^{-\mu b} 2\lambda^2 e^{-\lambda(a+b)} \, da \, db$$

$$= \frac{2\lambda\mu(\lambda^2 + 4\lambda\mu + \mu^2)}{(2\lambda + \mu)(\lambda + 2\mu)(\lambda + \mu)^2}.$$

Therefore,

$$\mathrm{PCS}^{\mathrm{avc}} = \frac{\mu(\lambda^2 + 6\lambda\mu + 2\mu^2)}{(2\lambda + \mu)(\lambda + 2\mu)(\lambda + \mu)}.$$

Given expressions for both $\mathrm{PCS}^{\mathrm{bem}}$ and $\mathrm{PCS}^{\mathrm{avc}}$, we can find the increase in PCS with AVC as

$$\Delta\mathrm{PCS} = \mathrm{PCS}^{\mathrm{avc}} - \mathrm{PCS}^{\mathrm{bem}}$$

$$= \frac{\lambda\mu(\mu - \lambda)}{(2\lambda + \mu)(\lambda + 2\mu)(\lambda + \mu)} > 0. \tag{2.7}$$

The $(\mu - \lambda) > 0$ term in Equation (2.7) shows that when X is the best population, AVC always shows an improvement in PCS over BEM.

Similar calculations for $k = 2$ and $v = 3$ result in

$$PCS^{\mathrm{bem}} = \frac{\mu^2(3\lambda + \mu)}{(\lambda + \mu)^3}$$

$$PCS^{\mathrm{avc}} = \frac{\mu^2(20\lambda^5 + 159\lambda^4\mu + 344\lambda^3\mu^2 + 273\lambda^2\mu^3 + 92\lambda\mu^4 + 12\mu^5)}{(3\lambda + 2\mu)(2\lambda + 3\mu)(2\lambda + \mu)(\lambda + 2\mu)(\lambda + \mu)^3}$$

19

and

$$\Delta PCS = \frac{\lambda^2 \mu^2 (16\lambda^2 + 37\lambda\mu + 16\mu^2)(\mu - \lambda)}{(3\lambda + 2\mu)(2\lambda + 3\mu)(2\lambda + \mu)(\lambda + 2\mu)(\lambda + \mu)^3} > 0.$$

Also for $k = 3$ and $v = 2$ we obtain

$$PCS^{\text{bem}} = \frac{2\mu^2}{(\lambda + 2\mu)(\lambda + \mu)}$$

$$PCS^{\text{avc}} = \frac{2\mu^3(3\lambda^4 + 100\lambda^3\mu + 267\lambda^2\mu^2 + 194\lambda\mu^3 + 36\mu^4)}{(2\lambda + 3\mu)(2\lambda + \mu)(\lambda + 4\mu)(\lambda + 3\mu)(\lambda + 2\mu)(\lambda + \mu)^2}$$

and

$$\Delta PCS = \frac{2\lambda\mu^2(4\lambda^3 + 41\lambda^2\mu + 84\lambda\mu^2 + 41\mu^3)(\mu - \lambda)}{(2\lambda + 3\mu)(2\lambda + \mu)(\lambda + 4\mu)(\lambda + 3\mu)(\lambda + 2\mu)(\lambda + \mu)^2} > 0.$$

Each of the expressions above for the difference in PCS between AVC and BEM has a $(\mu - \lambda) > 0$ term, implying an increase in PCS with AVC.

2.4.3 Continuous Uniform

Consider $k = 2$ and $v = 2$ where X and O come from continuous uniform distributions. Specifically $X \sim U(0, B)$ and $O \sim U(0, A)$, where $0 < A < B$. Then we have $p_{[2]} = \Pr\{X > O\} = (2B - A)/2B$ and $p_{[1]} = \Pr\{X < O\} = A/2B$. From Equation (2.5) we obtain

$$\text{PCS}^{\text{bem}} = \frac{2B - A}{2B}.$$

The joint distribution of $(X_{[1]}, X_{[2]})$ is $f_{12}(g, h) = 2/B^2$. Then using Equation (2.6) we find

$$PCS^{\text{avc}} = \frac{A^2 + 2AB - 6B^2}{6B^2}.$$

20

As we did for our exponential example, we can find the improvement in PCS with AVC is

$$\Delta \text{PCS} = \frac{A(B-A)}{6B} > 0.$$

Similar calculations for $k = 2$ and $v = 3$ result in

$$\Delta PCS = \frac{3A^2(B-A)}{20B^3} > 0.$$

Also for $k = 3$ and $v = 2$ we obtain

$$\Delta PCS = \frac{7A(B-A)}{15B^2} > 0.$$

Each of the expressions above for the difference in PCS between AVC and BEM has a $(B-A) > 0$ term, again indicating an improvement in PCS with AVC.

2.4.4   Bernoulli

As an illustration of how AVC compares to BEM for discrete distributions, let $X \sim \text{Bern}(p_x)$ and $O \sim \text{Bern}(p_o)$ with $p_x > p_o$. For $k = 2$, $v = 2$, we have $p_{[2]} = \text{Pr}\{X > O\} = (p_x + 1 - p_o)/2$ and $p_{[1]} = \text{Pr}\{X < O\} = (p_o + 1 - p_x)/2$. From Equation (2.5) we obtain

$$\text{PCS}^{\text{bem}} = \frac{1}{2}(p_x + 1 - p_o).$$

PCS^{avc} can be calculated in a similar fashion as we did for our exponential and continuous uniform examples,

$$\text{PCS}^{\text{avc}} = \frac{1}{4}(p_o^2 - p_x^2) + \frac{1}{2}p_x p_o(p_x - p_o) + \frac{3}{4}(p_x - p_o) + \frac{1}{2}.$$

Then calculating the difference in PCS,

$$\Delta\text{PCS} = \frac{1}{4}(1 + 2p_x p_o - p_x - p_o)(p_x - p_o) > 0. \tag{2.8}$$

We again see a term, $(p_x - p_o) > 0$ in Equation (2.8), which illustrates an improvement in PCS with AVC when X is the better population.

2.4.5 Effect of Population Distributions on AVC

Our analytical results are presented for single values of k and v for each of the distributions. To compare PCS^{avc} across the distributions, we use a fixed value of k and v and vary $\theta = p_{[k]}/p_{[k-1]}$. We display these results using the ratio of improvement $\text{PCS}^{\text{avc}}/\text{PCS}^{\text{bem}}$ plotted against θ. Recall that PCS^{bem} is distribution independent.

Figure 2.1 shows results for exponential, continuous uniform, and Bernoulli populations at $k = 2$, $v = 2$. The exponential and uniform curves depend only on θ and approach 1 as θ approaches infinity. However, for the Bernoulli results, we must specify a value for either p_x or p_o, which forces an upper limit on the value of θ as p_x approaches 1 or as p_o approaches 0. We set $p_x = 7/8$ and then $\lim_{p_o \to 0} \theta = 15$. Figures 2.2 and 2.3 display results for exponential and continuous uniform populations at $k = 2$, $v = 3$ and $k = 3$, $v = 2$ respectively.

There are some significant results to gleen from Figures 2.1–2.3. First, notice the spread between the results for the different populations is quite small (a maximum of roughly 5% for $k = 3$, $v = 2$, and on the order of 1% for $k = 2$, $v = 2$ and $k = 2$, $v = 3$), and this spread drops off as θ approaches one. This indicates that
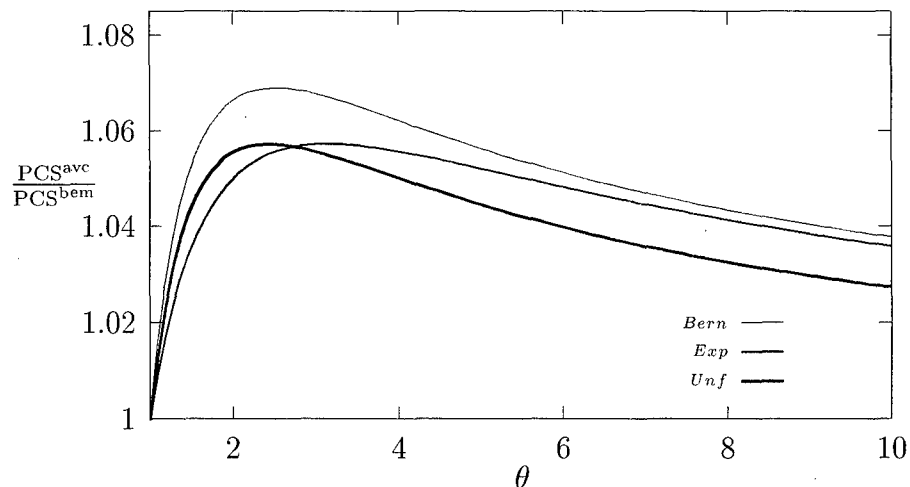
Figure 2.1: Exponential, Uniform and Bernoulli Populations: $k = 2$, $v = 2$, $p_x = \frac{7}{8}$

as the difference between the best and the next system gets smaller, the distributional dependence becomes weaker. Also notice that the maximum improvement with AVC, indicated by the peaks in the curves, occurs roughly over a range of θ between 1.5 and 3 for all the distributions. This covers most of the practical range of θ included in standard tables and used by experimentors.

The weak distributional dependence of $\mathrm{PCS^{avc}}$, along with the difficulty of computing $\mathrm{PCS^{avc}}$ for small k and v, motivates the following large-sample approximation (LSA).
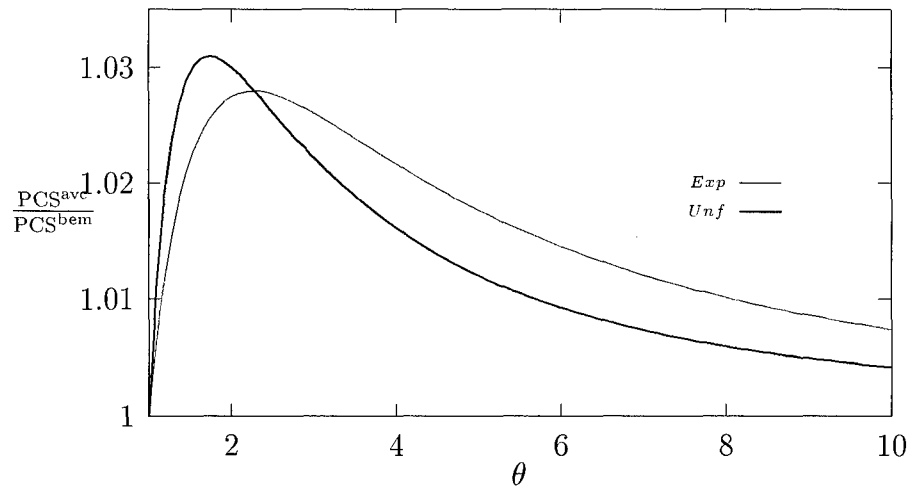
Figure 2.2: Exponential and Uniform Populations: $k = 2$, $v = 3$

2.5   Large-Sample Approximation

The results presented so far for small k and v show that $\mathrm{PCS}^{\mathrm{avc}}$ is weakly distribution dependent. By redefining our $\mathrm{PCS}^{\mathrm{bem}}$ and $\mathrm{PCS}^{\mathrm{avc}}$ in terms of point estimators for each of the individual system success probabilities, we arrive at distribution-free results as the sample size goes to infinity.

2.5.1   Preliminaries

Using our previous notation we have

$$p_j = Pr\{X_{ji} > X_{\ell i}, \forall \ell \neq j\}.$$
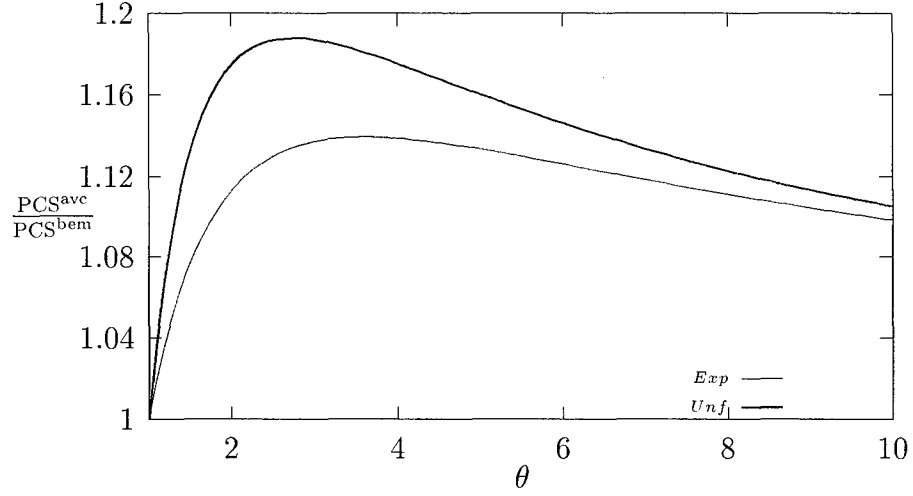
Figure 2.3: Exponential and Uniform Populations: $k = 3$, $v = 2$

Let the distribution of X_{ji} depend upon the sample size, $X_j \sim F_j^{(v)}$. We construct the $F_j^{(v)}$ such that the $F_j^{(v)}$ converge to a common distribution, F, for all j as v approaches infinity and for finite v

$$\Pr\{X_{ji} > X_{j\ell} \mid \text{sample size } v\} = p_j(v)$$
$$= \begin{cases} \frac{1}{k} + \frac{(k-1)\delta}{\sqrt{v}} & j = 1 \\ \frac{1}{k} - \frac{\delta}{\sqrt{v}} & j \neq 1. \end{cases} \tag{2.9}$$

Define

$$Y_j(v) = \text{number of wins for system } j \text{ under BEM with sample size v}$$

$$Z_j(v) = \text{number of wins for system } j \text{ under AVC with sample size v}$$

which gives us point estimators

$$\widehat{p}_j(v) = \frac{Y_j(v)}{v}$$

25

$$\bar{p}_j(v) = \frac{Z_j(v)}{v^k}.$$

So our BEM estimators are denoted by $\hat{p}_j$ and our AVC estimators by $\bar{p}_j$. Under (2.9) Population 1 is the best. Notice that (ignoring the asymptotically vanishing probability of a tie)

$$\text{PCS}^{\text{bem}} = \Pr\{\hat{p}_1 > \hat{p}_j, \forall j \neq 1\}$$

$$\text{PCS}^{\text{avc}} = \Pr\{\bar{p}_1 > \bar{p}_j, \forall j \neq 1\}.$$

Our approach is based on the fact that standardized versions of $\hat{p}$ and $\bar{p}$ are asymptotically multivariate normal (MVN). However, when the distributions are fixed then as the sample size increases, both PCS^{bem} and PCS^{avc} approach 1, masking the differences between the two procedures. To eliminate this effect and isolate the improvement with AVC, we simultaneously let the ratio of the differences between the populations approach 1 at the canonical rate of $1/\sqrt{v}$, as shown in (2.9).

2.5.2 BEM Estimators

Consider the asymptotic behavior of PCS^{bem} as the number of vectors, v, goes to infinity. Our approach is structured around a result presented by Lehmann (1986, pp. 478-479) which we state below as a lemma.

Lemma 2.1 *(Asymptotic Distribution of Standardized BEM Estimators) Let* $\mathbf{Y} = (Y_1(v), Y_2(v), \ldots, Y_k(v))$ *be distributed as a multinomial random variable with parameters* v *and* $\mathbf{p} = (p_1(v), p_2(v), \ldots, p_k(v))$. *Define* $p_j(v)$ *as in equation (2.9).*

Let

$$A_j(v) = \frac{Y_j(v) - v/k}{\sqrt{v}} \qquad j = 1, 2, \ldots, k.$$

Then as $v \to \infty$

$$\begin{pmatrix} A_2(v) \\ \vdots \\ A_k(v) \end{pmatrix} \Longrightarrow$$

$$MVN\left[\begin{pmatrix} -\delta \\ \vdots \\ -\delta \end{pmatrix}, \begin{pmatrix} 1/k(1-1/k) & -1/k(1/k) & \cdots & -1/k(1/k) \\ -1/k(1/k) & 1/k(1-1/k) & \cdots & -1/k(1/k) \\ \vdots & \vdots & \ddots & \vdots \\ -1/k(1/k) & -1/k(1/k) & \cdots & 1/k(1-1/k) \end{pmatrix} \right].$$

Proof: See Lehmann (1986).

Lemma 2.1 is critical to proving the following theorem which we later use to equate asymptotic PCS$^{\text{bem}}$ with asymptotic PCS$^{\text{avc}}$.

Theorem 2.1 *(Asymptotic PCS^{bem})*

Let $\mathbf{Y} = (Y_1(v), Y_2(v), \ldots, Y_k(v))$ be distributed as a multinomial random variable with parameters v and $\mathbf{p} = (p_1(v), p_2(v), \ldots, p_k(v))$, with p_j defined as in equation (2.9). Then

$$\overrightarrow{\text{PCS}}^{\text{bem}} \equiv \lim_{v \to \infty} \Pr\{Y_1(v) > Y_j(v)\} = \Pr\left\{ \max_{\ell=2,\ldots,k} Q_\ell < \frac{k\delta}{\sqrt{2/k}} \right\} \quad (2.10)$$

where

$$\begin{pmatrix} Q_2 \\ \vdots \\ Q_k \end{pmatrix} \sim MVN\left[\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1/2 \\ & \ddots & \\ 1/2 & & 1 \end{pmatrix} \right].$$

Proof: See Appendix A.

27

2.5.3 AVC Estimators

Consider the asymptotic behavior of PCS$^{\text{avc}}$ as the number of vectors, v, goes to infinity. Our approach is structured around a result presented by Lehmann (1963, pp. 964-965) and also in Randles and Wolfe (1979, p. 107). We state this result below as Lemma 2.2.

We notice that our AVC estimator is a k-sample U-statistic, where $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ represents our parameter for the system success probabilities. From Randles and Wolfe (1979, p. 104), we say $\mathbf{p}$ is estimable of degree $(1, \ldots, 1)$ for distributions $(F_1, \ldots, F_k)$ in some family $\mathcal{F}$, if $(1, \ldots, 1)$ are the smallest sample sizes for which there exists an unbiased estimator of $\mathbf{p}$ for every $(F_1, \ldots, F_k) \in \mathcal{F}$. Formally stated we have

$$E_{(F_1,\ldots,F_k)}[h^{(j)}(X_{11}, \ldots, X_{k1})] = p_j$$

for $j = 1, \ldots, k$ and we define our k-sample symmetric kernel $h^{(j)}(\cdot)$ as

$$h^{(j)} = \prod_{\ell=1;\ell \neq j}^{k} \phi(X_{ji} - X_{\ell i}). \tag{2.11}$$

So we have $h^{(j)} = 1$ if the observation from the jth system is the largest in any vector comparison across all systems. Substituting our kernel from (2.11) into (2.2) and dividing by v^k, we have presented our $\bar{p}_j$, $j = 1, 2, \ldots, k$ as k-sample U-statistics. We can then state the following lemma, where some of the notation has been simplified to match the context of our problem.

Lemma 2.2 *(Asymptotic Distribution of AVC Estimators) Let $\bar{p}_1, \ldots, \bar{p}_k$ be k-sample ($k \geq 1$) U-statistics, with $\bar{p}_i$ corresponding to a parameter p_i of degree $(1, \ldots, 1)$ and symmetric kernel $h^{(i)}(\cdot)$, for $i = 1, \ldots, k$. Let $N = kv$, where v is*

28

the sample size from each of k populations. Then the joint limiting distribution of

$$
\begin{pmatrix} \sqrt{N}(\bar{p}_1 - p_1) \\ \vdots \\ \sqrt{N}(\bar{p}_k - p_k) \end{pmatrix} \implies MVN \left[\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \Sigma = \left(\sigma^{(a,b)} \right) \right]
$$

where

$$
\sigma^{(a,b)} = \sum_{i=1}^{k} \frac{1}{\lambda_i} \xi_i^{(a,b)} \tag{2.12}
$$

for $\lambda_i = \lim_{N \to \infty} v/N$. The quantities $\xi_i^{(a,b)}$ are given by

$$
\begin{aligned}
\xi_i^{(a,b)} &= Cov\left[H_{i1}^{(a)}, H_{i2}^{(b)} \right] \\
&= E\left[H_{i1}^{(a)} H_{i2}^{(b)} \right] - p_a p_b
\end{aligned}
$$

where

$$
\begin{aligned}
H_{i1}^{(a)} &= h^{(a)}(X_{1\alpha_1}, X_{2\alpha_2}, \dots, X_{k\alpha_k}) \\
H_{i2}^{(b)} &= h^{(b)}(X_{1\beta_1}, X_{2\beta_2}, \dots, X_{k\beta_k})
\end{aligned}
$$

and the sets $(\alpha_1, \alpha_2, \dots, \alpha_i, \dots, \alpha_k)$ and $(\beta_1, \beta_2, \dots, \beta_i, \dots, \beta_k)$ have only the ith element in common, where the elements in each set represent positive integers and $a, b \in \{1, \dots, k\}$.

Proof: See Lehmann (1963).

Proceeding in much the same manner as we did in moving from Lemma 2.1 to Theorem 2.1, we can define PCS$^{\text{avc}}$ as a probability statement involving a function of k and the maximum of $(k-1)$ random variables created by subtracting one of the standardized $\bar{p}_j$s from each of the remaining standardized $\bar{p}_i$s, $i \neq j$. These random variables are the $Q_2, Q_3, \dots, Q_k$ in the following theorem.

29

Theorem 2.2 *(Asymptotic* PCS$^{\mathrm{avc}}$*)*

Let $\bar{p}_1, \ldots, \bar{p}_k$ be k-sample ($k \geq 1$) U-statistics, with $\bar{p}_i$ corresponding to a parameter p_i of degree $(1, \ldots, 1)$ and symmetric kernel $h^{(i)}(\cdot)$, for $i = 1, \ldots, k$. Let $N = kv$, where v is the sample size from each of k populations. Let $\mathbf{p}(v) = (p_1(v), p_2(v), \ldots, p_k(v))$, with $p_j(v)$ defined as in equation (2.9). Then under our model with $F_j^{(v)} \longrightarrow F$

$$\overrightarrow{\mathrm{PCS}}^{\mathrm{avc}} \equiv \lim_{v \to \infty} \Pr\{\bar{p}_1(v) > \bar{p}_j(v)\} = \Pr\left\{\max_{\ell=2,\ldots,k} Q_\ell < \frac{k\delta}{\sqrt{2/(2k-1)}}\right\} \quad (2.13)$$

where

$$\begin{pmatrix} Q_2 \\ \vdots \\ Q_k \end{pmatrix} \sim MVN\left[\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & & 1/2 \\ & \ddots & \\ 1/2 & & 1 \end{pmatrix}\right].$$

Proof: See Appendix A.

2.5.4   Combining BEM and AVC Results

Let $\overrightarrow{\mathrm{PCS}}^{\mathrm{avc}}$ and $\overrightarrow{\mathrm{PCS}}^{\mathrm{bem}}$ represent the asymptotic PCS for AVC and BEM respectively, under the setup described in §2.5.2 and §2.5.3. Combining the results from Equations (2.10) and (2.13) we have

$$\overrightarrow{\mathrm{PCS}}^{\mathrm{avc}} \geq \overrightarrow{\mathrm{PCS}}^{\mathrm{bem}}$$

since

$$\frac{k\delta^{\mathrm{avc}}}{\sqrt{2/(2k-1)}} \geq \frac{k\delta^{\mathrm{bem}}}{\sqrt{2/k}}$$

for $k \geq 2$ with $\delta^{\mathrm{avc}} = \delta^{bem}$. Then we have

$$\overrightarrow{\mathrm{PCS}}^{\mathrm{avc}} = \overrightarrow{\mathrm{PCS}}^{\mathrm{bem}} \quad (2.14)$$

if and only if

$$\frac{k\delta^{\text{avc}}}{\sqrt{2/(2k-1)}} = \frac{k\delta^{\text{bem}}}{\sqrt{2/k}}.$$

Solving for δ^{avc} we have

$$\delta^{\text{avc}} = \delta^{\text{bem}}\sqrt{k/(2k-1)}. \tag{2.15}$$

We use δ^{bem} and θ^{bem} (δ^{avc} and θ^{avc}) to represent the difference, $p_{[k]} - p_{[k-1]}$, or the ratio, $p_{[k]}/p_{[k-1]}$, respectively associated with BEM (AVC) calculations. Our goal is to use the relationship between δ^{avc} and δ^{bem} to define a relationship between θ^{avc} and θ^{bem} that also guarantees (2.14). This will allow us to use BEM calculations to approximate AVC results.

Under the LFC for BEM, Equation (2.1), we have

$$\begin{aligned} p_{[k]} - p_{[k-1]} &= \frac{\theta^*}{\theta^* + k - 1} - \frac{1}{\theta^* + k - 1} \\ &= \frac{\theta^* - 1}{\theta^* + k - 1}. \end{aligned} \tag{2.16}$$

To relate this difference in Equation (2.16) to our asymptotic analysis, we define the sample-size standardized difference as

$$\begin{aligned} \lim_{v \to \infty} \sqrt{v}(p_{[k]}(v) - p_{[k-1]}(v)) &= (k-1)\delta - (-\delta) \\ &= k\delta. \end{aligned} \tag{2.17}$$

Since (2.16) and (2.17) represent the same difference, we set

$$k\delta = \frac{\theta^* - 1}{\theta^* + k - 1} \tag{2.18}$$

and solving for θ^*

$$\theta^* = \frac{1 + (k-1)k\delta}{1 - k\delta}. \tag{2.19}$$

31

Equations (2.18) and (2.19) define a relationship between a ratio and a difference. We can proceed from here, using Equations (2.15) through (2.19), to relate θ^{avc} to θ^{bem} such that (2.14) holds. Summarizing the required steps we arrive at the following algorithm.

Algorithm 2.1 *(LSA for θ^{avc})*

1. *Express $k\delta^{avc}$ in terms of θ^{avc} using (2.18).*

2. *Rewrite $k\delta^{avc}$ from step 1 in terms of $k\delta^{bem}$ using (2.15).*

3. *Express θ^{bem} in terms of $k\delta^{bem}$ from step 2 using (2.19).*

Consider the following illustration. We have a problem with a specified (θ^*, k, P^*) where we want to find the minimum sample size required with AVC. To approximate the required sample size, v, for AVC we have

$$v^{avc}(\theta^{avc} = \theta^*, k, P^*) \approx v^{bem}(\theta^{bem}, k, P^*) \tag{2.20}$$

where v^{avc} and v^{bem} denote the v required for AVC or BEM respectively.

We can make this approximation since

$$\text{PCS}^{\text{bem}}\left(v^{bem}(\theta^{bem}, k, P^*)\right) \geq P^*$$

and from (2.14) we have

$$\overrightarrow{\text{PCS}}^{\text{avc}}(\theta^{avc}, k) = \overrightarrow{\text{PCS}}^{\text{bem}}(\theta^{bem}, k),$$

which leads us to

$$\text{PCS}^{\text{avc}}\left(v^{bem}(\theta^{bem}, k, P^*)\right) \approx P^*.$$

Continuing with our example we step through Algorithm 2.1. At step 1 we have

$$k\delta^{avc} = \frac{\theta^{avc} - 1}{\theta^{avc} + k - 1}.$$

At step 2 we have

$$k\delta^{bem} = \left(\frac{\theta^{avc} - 1}{\theta^{avc} + k - 1}\right)\sqrt{\frac{2k - 1}{k}}.$$

Then at step 3 we have our LSA for θ^{avc}

$$\theta^{bem} = \frac{1 + (k - 1)\left(\frac{\theta^{avc}-1}{\theta^{avc}+k-1}\right)\sqrt{\frac{2k-1}{k}}}{1 - \left(\frac{\theta^{avc}-1}{\theta^{avc}+k-1}\right)\sqrt{\frac{2k-1}{k}}}. \tag{2.21}$$

It will always be the case that $\theta^{bem} > \theta^*$.

If we want to estimate an equivalent θ^{avc} for $\theta^{bem} = \theta^*$, we make the following approximation,

$$v^{bem}(\theta^{bem} = \theta^*, k, P^*) \approx v^{avc}(\theta^{avc}, k, P^*), \tag{2.22}$$

using the same reasoning as for (2.20). Proceeding as in Algorithm 2.1, we start by expressing δ^{bem} in terms of δ^{avc} and end up writing θ^{avc} in terms of $k\delta^{avc}$ to obtain

$$\theta^{avc} = \frac{1 + (k - 1)\left(\frac{\theta^{bem}-1}{\theta^{bem}+k-1}\right)\sqrt{\frac{k}{2k-1}}}{1 - \left(\frac{\theta^{bem}-1}{\theta^{bem}+k-1}\right)\sqrt{\frac{k}{2k-1}}}. \tag{2.23}$$

It will always be the case that $\theta^{avc} < \theta^*$. This form of the approximation has little practical use. However, it does reflect another benefit of AVC in terms of a smaller θ, indicating the ability of AVC to discriminate smaller differences between the best and the next best system with the same value of v as BEM for θ^*. This advantage becomes important in a case where we need to detect as small a difference as possible with a fixed number of vector replications.

We provide conversions using both of these approximations for some common values of θ^* in Table 2.1 and present simulation results testing the robustness of the approximations in §2.7.

k	θ^*	$\theta^{bem} = \theta^*$ θ^{avc}	$\theta^{avc} = \theta^*$ θ^{bem}
2	1.2	1.1604	1.2506
	2.0	1.7479	2.3798
3	1.2	1.1526	1.2633
	2.0	1.7205	2.4297
4	1.2	1.1494	1.2689
	2.0	1.7124	2.4390
5	1.2	1.1476	1.2720
	2.0	1.7092	2.4400
10	1.2	1.1443	1.2778
	2.0	1.7061	2.4326

Table 2.1: Equivalent θ Values using LSA

2.6 Empirical Results

In order to allow easy comparison with available BEM results, we selected population distributions for our simulations that allowed us to control the value of θ^*. These distributions are the exponential, continuous uniform, and the Bernoulli presented in §2.4. In addition, to consider a less peaked continuous distribution without the restricted range of the continuous uniform, we looked at a set of gamma distributions with shape parameter $\alpha = 3$.

As in our analytical results, we consider population distributions that belong to the same parametric family. We arbitrarily select π_1 as the best population and

	θ	
k	1.2	2.0
2	1.2000	2.0000
3	1.1589	1.7808
4	1.1368	1.6632
5	1.1227	1.5885

Table 2.2: Exponential: Values of μ in LFC with $\lambda = 1$

the remaining populations are identically distributed. Let X_j represent a random observation from π_j; $j = 1, 2, \ldots, k$. We have

$$\Pr\{\text{Best Population Wins}\} = \Pr\{X_1 > \max(X_2, \ldots, X_k)\}.$$

We then define

$$\theta = \frac{\Pr\{X_1 > \max(X_2, \ldots, X_k)\}}{(1 - \Pr\{X_1 > \max(X_2, \ldots, X_k)\})/(k-1)}.$$

By setting $\theta = \theta^*$, we can then fix one or more parameters for one of the distributions and solve for the remaining parameter to carry out our simulations at a given θ^*. Tables 2.2–2.5 list parameters for $\theta = 1.2$ and $\theta = 2.0$ with $k = 2, 3, \ldots, 5$ for each of our four distributions.

Our simulation consists of the following steps.

1. Model all systems using the same distribution family, with system 1 arbitrarily the best, and all remaining systems identically distributed such that $\theta = \theta^*$. Initialize PCS^{bem} and PCS^{avc} to 0 and set $v = 2$.

2. Generate a set of v random vector replications, where each replication contains one observation for each of the k systems.

	θ	
k	1.2	2.0
2	1.1021	1.4442
3	1.0855	1.3751
4	1.0760	1.3340
5	1.0698	1.3061

Table 2.3: Gamma: Values of β_x in LFC with $\alpha_x = \alpha_o = 3$, $\beta_o = 1$

	θ	
k	1.2	2.0
2	1.1000	1.5000
3	1.0667	1.3333
4	1.0500	1.2500
5	1.0400	1.2000

Table 2.4: Continuous Uniform: Values of B in LFC with $A = 1$

	θ	
k	1.2	2.0
2	0.4091	0.1667
3	0.4208	0.2192
4	0.4250	0.2426
5	0.4264	0.2545

Table 2.5: Bernoulli: Values of p_o in LFC with $p_x = .5$

36

3. For BEM, group the observations across systems by vector replication and count up the number of wins for each system. These are our Y_j, $j = 1, 2, \ldots, k$.

4. For AVC, form the v^k pseudo-replications from the v vector replications and count the number of wins for each system. These are our Z_j, $j = 1, 2, \ldots, k$.

5. If Y_1 (BEM count associated with the best system) is larger than Y_j, $j = 2, 3, \ldots, k$, increase PCS$^{\text{bem}}$ by 1. If Y_1 ties for the largest count with t other systems, $t = 1, 2, \ldots, k-1$, increase PCS$^{\text{bem}}$ by $1/(t+1)$. If $Y_1 < Y_j$, for any j, $j = 2, 3, \ldots, k$, do not increase PCS$^{\text{bem}}$.

6. If Z_1 (AVC count associated with the best system) is larger than Z_j, $j = 2, 3, \ldots, k$, increase PCS$^{\text{avc}}$ by 1. If Z_1 ties for the largest count with t other systems, $t = 1, 2, \ldots, k-1$, increase PCS$^{\text{avc}}$ by $1/(t+1)$. If $Z_1 < Z_j$, for any j, $j = 2, 3, \ldots, k$, do not increase PCS$^{\text{avc}}$.

7. Repeat steps 2–6 for M macro-replications. Compute PCS$^{\text{bem}}$ = PCS$^{\text{bem}}/M$ and PCS$^{\text{avc}}$ = PCS$^{\text{avc}}/M$.

8. Increase v and repeat steps 2–7.

Using parameter values from Tables 2.2–2.5, we estimated PCS$^{\text{bem}}$ and PCS$^{\text{avc}}$ using the simulation described above for $k = 2, 3, \ldots, 5$ populations out to $v = 50$ vectors for each of the three continuous distributions at $\theta = 1.2$ and 2.0. Results were also obtained for $k = 10$ for exponential populations. Due to limited computer time, Bernoulli distributions were only simulated for $k = 2$ and 3 populations at $\theta = 1.2$. All simulation results are for $M = 100,000$ macro-replications using a

separate random number stream for each population, but common random numbers across distributions. Since the exponential and uniform random variates each require a single random number, results for these distributions have synchronized random variates for all of the data generated. This is not the case for the gamma random variates since an acceptance-rejection method was used (Law and Kelton 1991). The Bernoulli random variates are synchronized; however, because of the possibility of ties among the systems within a vector replication, an additional random number stream is used a random number of times to break the ties for the PCS calculations. Standard errors for the PCS values are on the order of 0.0015. More complete results are available in Appendix B.

Table 2.6 list results for each of our distributions out to $v = 50$ vectors for $k = 2$ populations at $\theta = 1.2$. The PCS^{bem} column is from simulations using exponential populations. The difference in the PCS^{avc} values among the continuous distributions is generally found in the third decimal place. However, we see a more significant difference between the Bernoulli PCS^{avc} and any of the continuous PCS^{avc} values. Figure 2.4 demonstrates the distributional dependence of PCS^{avc} for exponential and Bernoulli populations. We also notice significant improvement in Table 2.6 with PCS^{avc} over PCS^{bem} for all of the distributions. Figure 2.5 illustrates the improvement with PCS^{avc} over PCS^{bem} for $k = 2$ to 5 exponential populations. Looking closely at Figure 2.5, the spread between PCS^{avc} and PCS^{bem} appears to be increasing slightly as k increases. This is most readily apparent when comparing the $k = 2$ results to the $k = 3$ results. It is also apparent from both

		PCS$^{\text{avc}}$			
v	PCS$^{\text{bem}}$	Exponential	Uniform	Gamma	Bernoulli
2	0.5430	0.5532	0.5579	0.5532	0.5565
3	0.5641	0.5667	0.5712	0.5667	0.5790
4	0.5656	0.5779	0.5821	0.5762	0.5919
5	0.5826	0.5873	0.5913	0.5853	0.6037
6	0.5835	0.5950	0.5990	0.5940	0.6148
7	0.5971	0.6032	0.6074	0.6019	0.6251
8	0.5973	0.6101	0.6140	0.6098	0.6356
9	0.6102	0.6172	0.6201	0.6164	0.6437
10	0.6099	0.6232	0.6261	0.6238	0.6513
11	0.6190	0.6297	0.6317	0.6295	0.6586
12	0.6193	0.6351	0.6364	0.6351	0.6660
13	0.6287	0.6406	0.6422	0.6409	0.6726
14	0.6290	0.6463	0.6470	0.6460	0.6781
15	0.6372	0.6507	0.6523	0.6518	0.6856
16	0.6380	0.6553	0.6563	0.6564	0.6900
17	0.6460	0.6598	0.6611	0.6612	0.6970
18	0.6462	0.6638	0.6653	0.6662	0.7034
19	0.6548	0.6681	0.6700	0.6700	0.7085
20	0.6541	0.6730	0.6744	0.6731	0.7132
21	0.6620	0.6773	0.6780	0.6776	0.7180
22	0.6624	0.6812	0.6822	0.6813	0.7223
23	0.6705	0.6848	0.6862	0.6851	0.7275
24	0.6695	0.6880	0.6902	0.6888	0.7320
25	0.6760	0.6915	0.6940	0.6924	0.7363
26	0.6757	0.6950	0.6976	0.6957	0.7407
27	0.6818	0.6992	0.7011	0.6995	0.7454
28	0.6818	0.7020	0.7043	0.7030	0.7502
29	0.6882	0.7049	0.7073	0.7055	0.7548
30	0.6882	0.7078	0.7104	0.7088	0.7576
31	0.6935	0.7110	0.7138	0.7117	0.7613
32	0.6940	0.7145	0.7171	0.7144	0.7642
33	0.6992	0.7176	0.7199	0.7175	0.7675
34	0.6987	0.7206	0.7228	0.7200	0.7716
35	0.7049	0.7235	0.7257	0.7233	0.7746
36	0.7044	0.7261	0.7288	0.7258	0.7785
37	0.7096	0.7295	0.7314	0.7284	0.7818
38	0.7094	0.7324	0.7346	0.7314	0.7853
39	0.7140	0.7352	0.7377	0.7344	0.7880
40	0.7138	0.7378	0.7403	0.7374	0.7907
41	0.7195	0.7406	0.7433	0.7401	0.7928
42	0.7198	0.7427	0.7460	0.7429	0.7955
43	0.7246	0.7455	0.7483	0.7457	0.7988
44	0.7241	0.7479	0.7508	0.7480	0.8015
45	0.7282	0.7503	0.7528	0.7506	0.8040
46	0.7279	0.7528	0.7549	0.7529	0.8073
47	0.7319	0.7550	0.7570	0.7551	0.8098
48	0.7319	0.7567	0.7595	0.7572	0.8121
49	0.7357	0.7591	0.7618	0.7591	0.8155
50	0.7362	0.7614	0.7638	0.7612	0.8179

Table 2.6: PCS Results for $k = 2$ Populations with $\theta = 1.2$
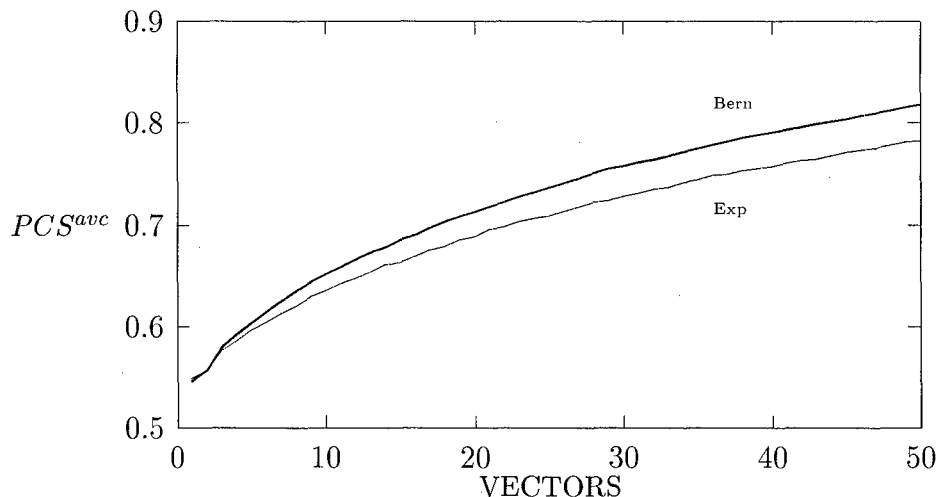
Figure 2.4: PCS$^{\text{avc}}$ for Exponential and Bernoulli Populations: $k = 2$, $\theta = 1.2$

Figures 2.4 and 2.5 that the spread between the PCS$^{\text{avc}}$ and PCS$^{\text{bem}}$ values widens as v increases. However, we know that as v approaches infinity both PCS$^{\text{avc}}$ and PCS$^{\text{bem}}$ approach 1, so this spread will eventually go to zero.

These results clearly show an improvement in PCS with AVC for all values of k and v considered, and also illustrate the weak dependence of PCS$^{\text{avc}}$ on the underlying population distributions.

2.7 Robustness of LSA

To check the accuracy of our LSA approximation, we performed a simulation study. The study covers a range of values for P^* (.75, .90, and .95) and θ^* (1.2 and 2.0) with exponential, continuous uniform and gamma distributions for $k = 2, 3, \ldots, 5$
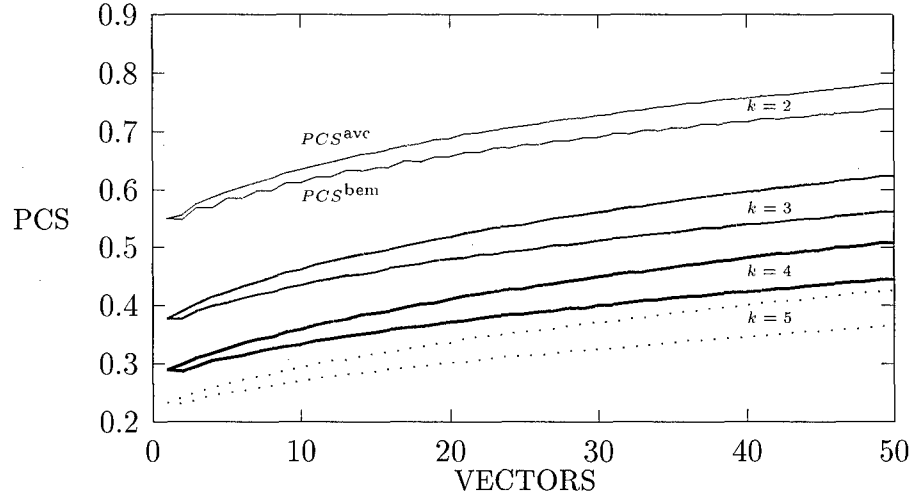
Figure 2.5: PCS for Exponential Populations, $\theta = 1.2$

populations and Bernoulli distributions for $k = 2, 3$. For the exponential and continuous uniform distributions, results are also presented for $k = 10$ at $\theta^* = 2.0$. Results are not available for all distributions at $\theta^* = 1.2$ for $k > 2$. This is due to the significant amount of computing time required to obtain these results due to the much larger number of replications required than for $\theta^* = 2.0$. We have included all results available for $\theta^* = 1.2$ in Tables 2.7 and 2.9.

We first consider the approximation in (2.20) and perform the following steps.

1. Select a k and θ^* and set $\theta^{avc} = \theta^*$. This indicates that we are interested in calculating AVC results at θ^*.

2. Solve for θ^{bem} using Equation (2.21).

41

3. The calculated value of θ^{bem} will not be in a standard BEM table. Use FORTRAN code developed by Goldsman (1995) to find $v^{bem}(\theta^{bem}, k, P^*)$ for $P^* = .75, .90, .95$. Denote these values as $v^{bem}(.75)$, $v^{bem}(.90)$, and $v^{bem}(.95)$, respectively.

4. Perform simulation runs to estimate PCS^{avc} values at k and $\theta^{avc} = \theta^*$ when using the v^{bem} values from step 3. We are looking for the following:

$$\text{PCS}^{avc}\left(v^{bem}(.75); (\theta^{avc}, k)\right) \approx .75$$

$$\text{PCS}^{avc}\left(v^{bem}(.90); (\theta^{avc}, k)\right) \approx .90$$

$$\text{PCS}^{avc}\left(v^{bem}(.95); (\theta^{avc}, k)\right) \approx .95.$$

Values estimated in step 4 above are reported in Tables 2.7 and 2.8. If our LSA is good, all PCS^{avc} values in Tables 2.7 and 2.8 should be close to the P^* listed at the top of the column in which they appear. The table values include the PCS^{avc} value and the associated standard error in parentheses. All simulation runs use the model described in §2.6 and are based on M=100,000 macro-replications for the three values of v found in step 3 above.

To illustrate how this approximation works for a numerical example, say we want to find $v^{avc}(\theta^{avc} = 1.2, k = 3, P^* = .90)$. Using (2.21) we obtain $\theta^{bem} = 1.2633$, and using FORTRAN code developed by Goldsman (1995), we find $v^{bem}(\theta^{bem} = 1.2633, k = 3, P^* = .90) = 264$. To show how good an approximation this provides for our specified v^{avc}, we simulate $M = 100,000$ macro-replications each containing 264 vector replications using $\theta^{avc} = 1.2$ with exponential populations, and obtain $\widehat{\text{PCS}}^{avc} = .8997$ with a standard error of 0.0009. So here our LSA is very good. These results are included in Table 2.7, where we see that the

42

k	θ^{bem}	Distribution	$P^* = .75$	$P^* = .90$	$P^* = .95$
2	1.2506	Exponential	.7502 (.0014)	.9012 (.0009)	.9510 (.0007)
2	1.2506	Gamma	.7481 (.0014)	.8994 (.0010)	.9493 (.0007)
2	1.2506	Uniform	.7499 (.0014)	.9010 (.0009)	.9506 (.0007)
2	1.2506	Bernoulli	.7856 (.0013)	.9318 (.0008)	.9723 (.0005)
3	1.2633	Exponential	.7470 (.0014)	.8997 (.0009)	.9490 (.0007)
3	1.2633	Gamma	.7446 (.0014)	.8994 (.0010)	.9482 (.0007)
3	1.2633	Uniform	.7474 (.0014)	.8989 (.0010)	.9481 (.0007)
4	1.2689	Exponential	.7469 (.0014)	.8964 (.0010)	.9485 (.0007)
4	1.2689	Gamma	.7462 (.0014)	.8984 (.0010)	.9489 (.0007)
4	1.2689	Uniform	.7469 (.0014)	.8948 (.0010)	.9467 (.0007)

Table 2.7: PCS Achieved for $\theta^{avc} = \theta^* = 1.2$ Using LSA

k	θ^{bem}	Distribution	$P^* = .75$	$P^* = .90$	$P^* = .95$
2	2.3798	Exponential	.7614 (.0013)	.8914 (.0010)	.9450 (.0007)
2	2.3798	Gamma	.7608 (.0013)	.8924 (.0010)	.9456 (.0007)
2	2.3798	Uniform	.7625 (.0013)	.8889 (.0010)	.9426 (.0007)
2	2.3798	Bernoulli	.7673 (.0013)	.9380 (.0008)	.9773 (.0005)
3	2.4297	Exponential	.7390 (.0014)	.8909 (.0010)	.9426 (.0007)
3	2.4297	Gamma	.7380 (.0014)	.8910 (.0010)	.9440 (.0007)
3	2.4297	Uniform	.7391 (.0014)	.8873 (.0010)	.9391 (.0007)
3	2.4297	Bernoulli	.7833 (.0013)	.9279 (.0008)	.9687 (.0007)
4	2.4390	Exponential	.7461 (.0014)	.8911 (.0010)	.9464 (.0007)
4	2.4390	Gamma	.7457 (.0014)	.8929 (.0010)	.9391 (.0008)
4	2.4390	Uniform	.7446 (.0014)	.8858 (.0010)	.9409 (.0007)
5	2.4400	Exponential	.7485 (.0014)	.8910 (.0010)	.9443 (.0007)
5	2.4400	Gamma	.7473 (.0014)	.8940 (.0010)	.9454 (.0007)
5	2.4400	Uniform	.7454 (.0014)	.8824 (.0010)	.9372 (.0007)

Table 2.8: PCS Achieved for $\theta^{avc} = \theta^* = 2.0$ Using LSA

results for all the distributions achieve the desired P^* to the second decimal place in almost all cases. In fact, we note that the Bernoulli results are significantly larger than P^* in many cases. The cases where we see more of a departure from P^* are for smaller values of P^* where v is typically less than 30.

We also notice that $v^{bem}(\theta^{bem} = 1.2, k = 3, P^* = .90) = 437$ (Bechhofer,Santner, and Goldsman 1995). Comparing this with our approximate $v^{avc}(\theta^{avc} = 1.2, k = 3, P^* = .90) = 264$, we see a nearly 40% reduction in the number of replications required with AVC.

To be complete we consider using the approximation in (2.22) and perform the following steps.

1. Select a k and θ^* and set $\theta^{bem} = \theta^*$. This indicates that we are interested in using BEM results at θ^*.

2. Solve for θ^{avc} using Equation (2.23).

3. Go to standard BEM tables, such as Table 8.1 in Bechhofer, Santner, and Goldsman (1995), and read off values for $v^{bem}(\theta^{bem}, k, P^*)$ for $P^* = .75, .90, .95$. Denote these values as $v^{bem}(.75)$, $v^{bem}(.90)$, and $v^{bem}(.95)$ respectively.

4. Perform simulation runs to obtain PCS^{avc} values at k and θ^{avc} from step 2 when using the v^{bem} values from step 3. We are looking for the following:

$$\text{PCS}^{avc}\left(v^{bem}(.75); (\theta^{avc}, k)\right) \approx .75$$
$$\text{PCS}^{avc}\left(v^{bem}(.90); (\theta^{avc}, k)\right) \approx .90$$
$$\text{PCS}^{avc}\left(v^{bem}(.95); (\theta^{avc}, k)\right) \approx .95.$$

44

Values estimated in step 4 above are reported in Tables 2.9 and 2.10. The table values include the PCS$^{\text{avc}}$ value and the associated standard error in parentheses. All simulation runs use the model described in §2.6 and are based on M=100,000 macro-replications for the three values of v found in step 3 above.

Following our previous numerical illustration, suppose we want to find $v^{bem}(\theta^{bem} = 1.2, k = 3, P^* = .90)$. Using (2.23) we obtain $\theta^{avc} = 1.1526$. We go to a standard BEM table, to find our specified $v^{bem} = 437$. As we did above, we test the robustness of our approximation by simulating 100,000 macro-replications each containing 437 vector replications using $\theta^{avc} = 1.1526$ with exponential populations, and obtain PCS$^{\text{avc}} = .8991$ with a standard error of 0.0010. Once again our LSA is very good. These results are presented with similar results for our other selected distributions in Tables 2.9 and 2.10.

The benefit from this form of the LSA is reflected by $\theta^{avc} < \theta^*$. This indicates that AVC can provide better discrimination between the systems at the same level of confidence and with the same data.

2.8 Conclusions

When trying to pick the best system out of k systems, there are many instances when this selection should be based on one-time performance rather than long-run average performance. Multinomial selection procedures provide a framework for defining such a problem, and Procedure BEM is the classical approach for solving it. Procedure AVC is an alternative approach designed to obtain a higher PCS by performing all possible comparisons across all systems for a given set of system

k	θ^{avc}	Distribution	$P^* = .75$	$P^* = .90$	$P^* = .95$
2	1.1604	Exponential	.7486 (.0014)	.8998 (.0009)	.9500 (.0007)
2	1.1604	Gamma	.7493 (.0014)	.9008 (.0009)	.9495 (.0007)
2	1.1604	Uniform	.7505 (.0013)	.9011 (.0009)	.9500 (.0007)
3	1.1526	Exponential	.7514 (.0014)	.8991 (.0010)	.9500 (.0007)
3	1.1526	Gamma	.7502 (.0014)	.8997 (.0010)	.9500 (.0007)
3	1.1526	Uniform	.7510 (.0013)	.8983 (.0009)	.9493 (.0007)
4	1.1494	Exponential	.7504 (.0014)	.8998 (.0010)	.9501 (.0007)
4	1.1494	Gamma	.7520 (.0014)	.9002 (.0009)	.9509 (.0007)
4	1.1494	Uniform	.7499 (.0014)	.8975 (.0010)	.9489 (.0007)

Table 2.9: PCS Achieved for $\theta^{bem} = \theta^* = 1.2$ Using LSA

k	θ^{avc}	Distribution	$P^* = .75$	$P^* = .90$	$P^* = .95$
2	1.7479	Exponential	.7692 (.0013)	.9021 (.0009)	.9471 (.0007)
2	1.7479	Gamma	.7702 (.0013)	.9037 (.0009)	.9477 (.0007)
2	1.7479	Uniform	.7703 (.0013)	.9013 (.0009)	.9453 (.0007)
2	1.7479	Bernoulli	.7950 (.0013)	.9373 (.0008)	.9726 (.0005)
3	1.7205	Exponential	.7471 (.0014)	.8971 (.0010)	.9460 (.0007)
3	1.7205	Gamma	.7456 (.0013)	.8975 (.0009)	.9464 (.0007)
3	1.7205	Uniform	.7459 (.0014)	.8933 (.0010)	.9432 (.0007)
3	1.7205	Bernoulli	.7911 (.0013)	.9332 (.0008)	.9701 (.0005)
4	1.7124	Exponential	.7419 (.0014)	.8944 (.0010)	.9461 (.0007)
4	1.7124	Gamma	.7418 (.0014)	.8956 (.0010)	.9464 (.0007)
4	1.7124	Uniform	.7403 (.0014)	.8888 (.0010)	.9414 (.0007)
5	1.7092	Exponential	.7458 (.0014)	.8950 (.0010)	.9474 (.0007)
5	1.7092	Gamma	.7452 (.0014)	.8976 (.0010)	.9478 (.0007)
5	1.7092	Uniform	.7412 (.0014)	.8881 (.0010)	.9410 (.0007)
10	1.7061	Exponential	.7576 (.0014)	.9024 (.0009)	.9516 (.0007)
10	1.7061	Uniform	.7476 (.0014)	.8891 (.0009)	.9411 (.0007)

Table 2.10: PCS Achieved for $\theta^{bem} = \theta^* = 2.0$ Using LSA

46

performance data. Construction of Procedure AVC closely follows that of BEM, allowing researchers to easily move from a standard approach to our new approach.

k	θ^*	P^*		
		.75	.90	.95
2	1.01	12171 (18371)	*	*
	1.05	509 (765)	1839 (2759)	3027 (4545)
	1.10	133 (201)	483 (723)	793 (1191)
	1.20	37 (55)	133 (199)	217 (327)
	2.00	3 (5)	9 (15)	15 (23)
3	1.05	1544 (2565)	3741(6211)	5526(9165)
	1.10	401 (666)	972 (1615)	1436 (2385)
	1.20	108 (181)	264 (437)	388 (645)
	2.00	7 (12)	17 (29)	25 (42)
4	1.20	187 (326)	398 (692)	565 (979)
	2.00	12 (20)	25 (43)	36 (61)
5	1.20	271 (486)	541 (964)	748 (1331)
	2.00	17 (29)	33 (58)	46 (81)

Table 2.11: Minimum Number of Vectors to Achieve P* for AVC (BEM)

From the simulation design point of view, AVC can also be used to our advantage by allowing us to plan a smaller number of replications to achieve a desired PCS, P^*. Table 2.8 presents comparisons of the minimum number of independent replications needed to achieve a given P^* for AVC and BEM. Values for BEM are taken from Bechhofer, Santner, and Goldsman (1995). The AVC values are obtained using our LSA in (2.21) with $\theta^{avc} = \theta^*$ to find θ^{bem} and then running an exact code for PCS^{bem} provided by Goldsman (1995) at $\theta = \theta^{bem}$. An asterik indicates that runs were not accomplished due to the large number of replications

($> 40,000$) required. As k increases, we see a more dramatic reduction in the number of vector observations needed with AVC to achieve the same P^*. So the advantages of AVC over BEM appear greater for more challenging MSPs.

References

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CHAPTER 3

POINT ESTIMATION FOR MULTINOMIAL SUCCESS PROBABILITIES

3.1 Introduction

Suppose we have $k \geq 2$ independent populations, denoted $\pi_1, \pi_2, \ldots, \pi_k$. In a simulation context each population is a simulated system. We consider the problem of estimating the probabilities associated with each system being the best system in a single comparison (trial) of simulated results for all of the systems.

Let X_{ji} represent the ith replication from system j of some performance measure. Each system $(\pi_j, \ j = 1, \ldots, k)$ has an unknown constant probability $(p_j, \ j = 1, \ldots, k)$ of having the largest value of the performance measure in any replication containing one observation from each system. We define the best system as the system most likely to have the largest performance measure (i.e., it wins) in any comparison across all systems. Such a comparison corresponds to a multinomial trial, where one and only one system can win in any given trial and the numbers of wins for the individual systems in v independent trials follows a multinomial

distribution. Our objective is to provide estimates for the unknown multinomial success probabilites, p_j, $j = 1, \ldots, k$.

Applications where a multinomial distribution is appropriate include selecting the best of a set of tactical or strategic military actions, where best means the maximum damage in a single strike. An example in the area of structural engineering is finding the design that performs best in a one-time catastrophic event, such as an earthquake. Simulation examples include selecting the schedule most likely to result in completing all jobs on time; selecting the investment portfolio most likely to provide the largest return; or selecting the computer system with the highest probability of completing a series of tasks without failure. Each of these applications involves the comparison of quantitative measures of performance among competing systems as opposed to comparing qualitative measures. For the type of application considered in this study, we require a quantitative measure of system performance for each system on each trial to be compared with the performance of other systems across any or all of the remaining trials.

Let $\mathbf{X_i} = (X_{1i}, X_{2i}, \ldots, X_{ki})$ represent the ith replication across all k systems. Let $Y_{ji} = 1$ if $X_{ji} > X_{\ell i}$, for $\ell = 1, 2, \ldots, k$, but $\ell \neq j$; and let $Y_{ji} = 0$ otherwise. In other words, $Y_{ji} = 1$ if X_{ji} is the largest observation in $\mathbf{X}_i$. In case of a tie for the largest value, we randomly select one of the tied populations as the best. Suppose that there are v independent replications across all systems, and let $Y_j = \sum_{i=1}^{v} Y_{ji}$ represent the number of times system j wins out of these v replications. Let $p_j = \Pr\{X_{ji} > X_{\ell i}, \forall \ell \neq j\}$ where $0 < p_j < 1$ and $\sum_{j=1}^{k} p_j = 1$. Then $\sum_{j=1}^{k} Y_j = v$

and the k-variate discrete random variable $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)$ follows a multinomial distribution with v trials and success probabilities $\mathbf{p} = (p_1, p_2, \ldots, p_k)$. The probability mass function for $\mathbf{Y}$ with parameters v and $\mathbf{p}$ is

$$\Pr\{Y_1 = y_1, Y_2 = y_2, \ldots, Y_k = y_k) = \frac{v!}{\prod_{j=1}^{k} y_j!} \prod_{j=1}^{k} p_j^{y_j}.$$

One set of point estimators for the multinomial success probabilities are the maximum likelihood estimates (MLEs) given by $\widehat{p}_j = Y_j/v$, $j = 1, 2, \ldots, k$.

Closely related to the estimation of the p_js is the problem of determining which of the systems has the largest probability of being the best system. This is known as the multinomial selection problem (MSP). The classical solution procedure for the MSP, Procedure BEM (Bechhofer, Elmaghraby, and Morse 1959), uses the multinomial cell counts, $\mathbf{Y}$, to select the best system. Some of the concepts and terminology used in the MSP carry over to our point estimation problem.

Due to convention and convenience when comparing simulated system responses, the responses are typically grouped by replication, corresponding to a trial in a physical experiment. Grouping system responses in this fashion is arbitrary and since our simulated responses are quantitative, we can compare any observation from one system with any observation from each of the remaining systems. This means that a single observation from system 1 can be grouped in a vector comparison with any one of the v observations from system 2, and with any one of the v observations from system 3, and so on, up to and including any one of the v observations from system k. Since there are v observations from system 1 as well, this gives us a total of v^k vector comparisons (trials) that can be formed with v independent observations from each of the k systems. We incorporate this

setup, which we call AVC for All Vector Comparisons, to construct new point estimators for the p_js. Our estimators turn out to be k-sample U-statistics (Randles and Wolfe 1979). By arbitrarily performing only the v vector comparisons where the observations for each system are from the same replication, we disregard the information available from the remaining $v^k - v$ comparisons.

Our results suggest some advantages of AVC estimators over the standard MLEs. First, we prove that the variance of the AVC estimators is no larger than the variance of the MLEs. For specific small-sample examples we demonstrate the magnitude of the variance reduction with AVC estimators for a fixed v. We show this analytically for small values of v and k. We also show that the variance reduction depends weakly on the distributions of the system performance measures. This fact, along with the difficulty of analytically evaluating the exact variance for AVC estimators for even small k and v, led us to a consider using the asymptotic variance as an approximation to the exact variance. An estimate of the asymptotic variance is used to form confidence intervals.

This paper is organized as follows: We first briefly address the MSP and introduce terminology used in our point estimation problem. We then present the MLEs and the associated variances for the p_js. Then we describe our new approach, AVC, which represents our point estimators as k-sample U-statistics. We present a proof for a variance reduction with AVC along with analytical results showing the magnitude of the variance reduction for small k and v covering different population distributions for the performance measures. Our asymptotic approximation for the variances of the AVC estimators is then presented. Empirical results follow and include the construction of confidence intervals.

3.2 Background

Because of the close relationship between the MSP and the estimation of multinomial success probabilities, we briefly address a classical solution procedure for the MSP and define some terminology we will be using throughout our discussion.

Bechhofer, Elmaghraby and Morse (1959) describe a single-stage procedure, Procedure BEM, for selecting the multinomial event (population or system) which has the largest success probability. Procedure BEM requires the specification of P^* (where $1/k < P^* < 1$), a minimum probability of correctly identifying the population with the largest success probability (i.e., the best population), and θ^* (where $1 < \theta^* < \infty$), the minimum ratio of the largest success probability to the second largest success probability that we want to be able to detect. The probability of correct selection (PCS) is a property of the selection procedure and provides no information about the values of the p_js. The procedure, as adapted to simulation, consists of the following steps:

Procedure 3.1 *(BEM)*

1. *For given k and θ^*, find the minimum value of v, denoted as v^*, that guarantees that the PCS is at least P^*.*

2. *Generate v^* independent replications for each population.*

3. *Compute $Y_j = \sum_{i=1}^{v^*} Y_{ji}$, for $j = 1, 2, \ldots, k$.*

4. Let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(k)}$ be the ranked sample counts from step 3. Select the population associated with the largest count, $Y_{(k)}$, as the best population. In case of a tie for the largest count, randomly select one of the tied populations as the best.

To determine the appropriate v^* in step 1, let $p_{[1]} \leq p_{[2]} \leq \cdots \leq p_{[k]}$ denote the ranked success probabilities for the k populations. Since only values of the ratio $\theta = p_{[k]}/p_{[k-1]}$ greater than or equal to θ^* are of interest, we are indifferent between the best and the next-best population for values of $\theta < \theta^*$. A procedure of this type is referred to as an *indifference-zone approach*. Select v^* as the minimum number of independent vector observations required to achieve a PCS greater than or equal to P^* whenever $\theta \geq \theta^*$.

We define the least favorable configuration (LFC) of $\mathbf{p} = (p_{[1]}, p_{[2]}, \ldots, p_{[k]})$ as the configuration where PCS is a minimum over all configurations with $\theta \geq \theta^*$ (Gibbons, Olkin, and Sobel 1977). If we obtain a PCS $\geq P^*$ with our selected v^* under the LFC, then a PCS of at least P^* can be guaranteed for *any* configuration of $\mathbf{p}$ with $\theta \geq \theta^*$. Keston and Morse (1959) prove that the LFC for BEM is given by

$$p_{[1]} = p_{[2]} = \cdots = p_{[k-1]} = \frac{1}{\theta^* + k - 1}$$
$$p_{[k]} = \frac{\theta^*}{\theta^* + k - 1}.$$

A well known set of estimators for our multinomial success probabilities are the MLEs given by

$$\widehat{p}_j = \frac{Y_j}{v}, \tag{3.1}$$

which represent the fraction of wins out of v replications for population j, $j = 1, \ldots, k$. It is well known that $\mathrm{E}[\widehat{p}_j] = p_j$ and

$$\mathrm{Var}(\widehat{p}_j) = \frac{p_j(1 - p_j)}{v}. \tag{3.2}$$

3.3 AVC Estimators

We propose a method to provide point estimators for the multinomial success probabilities with smaller variances than the corresponding MLEs using the same replications $\mathbf{X}_i, i = 1, 2, \ldots, v$. Rather than comparing the ith replication for each system with the ith replications of the other systems, consider a total of v^k pseudo-replications formed by associating each X_{ji} ($j = 1, 2, \ldots, k$; $i = 1, 2, \ldots, v$), with all possible combinations of the remaining $X_{\ell h}$ ($\ell = 1, 2, \ldots, k$; $\ell \neq j$; $h = 1, 2, \ldots, v$). Each such pseudo-replication contains one observation from each population. Notice that the v^k pseudo-replications include the v independent replications from which the pseudo-replications are formed.

Define

$$Z_j = \sum_{a_1=1}^{v} \sum_{a_2=1}^{v} \cdots \sum_{a_k=1}^{v} \prod_{\ell=1; \ell \neq j}^{k} \phi(X_{ja_j} - X_{\ell a_\ell}) \tag{3.3}$$

for $j = 1, 2, \ldots, k$ with

$$\phi(a) = \begin{cases} 1, & a > 0 \\ 0, & a < 0 \\ \text{randomly assign} \\ 0 \text{ or } 1, & a = 0. \end{cases}$$

Thus, Z_j represents the number of times out of v^k pseudo-replications that population π_j wins (ties broken randomly) and $\sum_{j=1}^{k} Z_j = v^k$.

As a specific illustration of how these pseudo-replications are formed, consider $k = 3$ systems with two observations generated for each. Our original replications and counts $(Y_j, \ j = 1, 2, 3)$ are then

$$
\begin{array}{ccc}
X_{11} & X_{21} & X_{31} \\
X_{12} & X_{22} & X_{32} \\
\hline
Y_1 & Y_2 & Y_3
\end{array}
$$

We will have a total of v^k ($2^3 = 8$) pseudo-replications including our two original replications when using AVC. These pseudo-replications and the associated counts $(Z_j, \ j = 1, 2, 3)$ for each system are

$$
\begin{array}{ccc}
X_{11} & X_{21} & X_{31} \\
X_{11} & X_{21} & X_{32} \\
X_{11} & X_{22} & X_{31} \\
X_{11} & X_{22} & X_{32} \\
X_{12} & X_{21} & X_{31} \\
X_{12} & X_{21} & X_{32} \\
X_{12} & X_{22} & X_{31} \\
X_{12} & X_{22} & X_{32} \\
\hline
Z_1 & Z_2 & Z_3
\end{array}
$$

Our new point estimators are then

$$
\bar{p}_j = \frac{Z_j}{v^k}, \tag{3.4}
$$

which represent the fraction of wins out of v^k pseudo-replications for population j. We refer to these estimators in (3.4) as AVC estimators, or to $\bar{\mathbf{p}} = (\bar{p}_1, \ldots, \bar{p}_k)$ collectively as our AVC estimator. Clearly $E[\bar{p}_j] = p_j$, but the $\text{Var}(\bar{p}_j)$ is more complex to calculate than $\text{Var}(\widehat{p}_j)$. To find the variances for the individual $\bar{p}_j$s, we represent our AVC estimator as a k-sample U-statistic, where $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ represents our parameters for the system success probabilities and $X_j \sim F_j$. From Randles and Wolfe (1979, p. 104), we say $\mathbf{p}$ is estimable of degree $(1, \ldots, 1)$ for

distributions $(F_1, \ldots, F_k)$ in some family $\mathcal{F}$, if $(1, \ldots, 1)$ are the smallest sample sizes from across all systems or populations for which there exists an unbiased estimator of $\mathbf{p}$ for every $(F_1, \ldots, F_k) \in \mathcal{F}$. Formally stated we have

$$E_{(F_1,\ldots,F_k)}[h^{(j)}(X_{11}, \ldots, X_{k1})] = p_j$$

for $j = 1, \ldots, k$ where we define our k-sample symmetric kernel $h^{(j)}(\cdot)$ as

$$h^{(j)} = \prod_{\ell=1;\ell\neq j}^{k} \phi(X_{ji} - X_{\ell i}). \tag{3.5}$$

So we have $h^{(j)} = 1$ if the observation from the jth system is the largest in any vector comparison across all systems. Substituting our kernel from (3.5) into (3.3) and dividing by v^k, we have presented our $\bar{p}_j$, $j = 1, 2, \ldots, k$ as k-sample U-statistics.

Using well-known results for U-statistics (Randles and Wolfe 1979), we develop the variance of a 2-sample U-statistic in general. First we have

$$\mathrm{E}_{(F_1,F_2)}[h(X_{1i}, X_{2i})] = p.$$

Then the covariance terms are

$$
\begin{aligned}
\xi_{1,0} &= \mathrm{Cov}\left[h(X_{1\alpha_1}, X_{2\alpha_2}),\ h(X_{1\alpha_1}, X_{2\beta_2})\right] \\
&= \mathrm{E}\left[h(X_{1\alpha_1}, X_{2\alpha_2})h(X_{1\alpha_1}, X_{2\beta_2})\right] - p^2 \\
\xi_{0,1} &= \mathrm{Cov}\left[h(X_{1\alpha_1}, X_{2\alpha_2}),\ h(X_{1\beta_1}, X_{2\alpha_2})\right] \\
&= \mathrm{E}\left[h(X_{1\alpha_1}, X_{2\alpha_2})h(X_{1\beta_1}, X_{2\alpha_2})\right] - p^2 \\
\xi_{1,1} &= \mathrm{Cov}\left[h(X_{1\alpha_1}, X_{2\alpha_2}),\ h(X_{1\alpha_1}, X_{2\alpha_2})\right] \\
&= \mathrm{E}\left[h(X_{1\alpha_1}, X_{2\alpha_2})h(X_{1\alpha_1}, X_{2\alpha_2})\right] - p^2
\end{aligned}
$$

where $\alpha_i \neq \beta_i$ and $\xi_{0,0} = 0$. In our specific context, suppose we want to estimate $p_1 = \mathrm{Pr}\{X_1 > X_2\}$. We use our kernel from equation (3.5) for $h^{(1)}(\cdot)$ and define

$$\xi_{1,0} = \mathrm{Pr}\{X_{1\alpha_1} > X_{2\alpha_2}; X_{1\alpha_1} > X_{2\beta_2}\} - p_1^2 \tag{3.6}$$

$$\xi_{0,1} \;=\; \Pr\{X_{1\alpha_1} > X_{2\alpha_2}; X_{1\beta_1} > X_{2\alpha_2}\} - p_1^2 \tag{3.7}$$

$$\xi_{1,1} \;=\; \Pr\{X_{1\alpha_1} > X_{2\alpha_2}; X_{1\alpha_1} > X_{2\alpha_2}\} - p_1^2 \tag{3.8}$$

where α_1, α_2, β_1, $\beta_2 \in \{1, \ldots, v\}$.

Then using a general equation for the variance of a 2-sample U-statistic, we can express the variance of our estimators as

$$\text{Var}(\bar{p}_j) = \frac{1}{v^2} \sum_{c=0}^{1} \sum_{d=0}^{1} \binom{v-1}{1-c} \binom{v-1}{1-d} \xi_{c,d}, \quad j = 1, \ldots, k. \tag{3.9}$$

3.4 Analytical Results

The following analytical study illustrates a number of important properties of our AVC estimator. First, we provide a general proof that the variance of our AVC estimator is no larger than the variance of the MLE. Then we quantify this variance reduction with AVC for specific cases. We also show that the variance reduction of AVC relative to MLE depends weakly on the underlying distribution for the X_{ji}. In addition we demonstrate the difficulty in obtaining analytical results for even a small number of populations and observations, and thus provide motivation for our asymptotic variance approximation.

3.4.1 Smaller Variance of the AVC Estimator

To show that AVC provides a point estimator with a smaller variance than MLE, suppose we have simulation results for k systems with v observations from each system as shown.

$$\begin{array}{cccc} X_{11} & X_{21} & \cdots & X_{k1} \\ X_{12} & X_{22} & \cdots & X_{k2} \\ \vdots & \vdots & & \vdots \\ \underline{X_{1v}} & \underline{X_{2v}} & \cdots & \underline{X_{kv}} \\ & & & \\ \widehat{F}_1 & \widehat{F}_2 & \cdots & \widehat{F}_k. \end{array}$$

Let $\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_k$ represent the empirical cumulative distribution function (cdf) for each population and suppose we wish to estimate p_1. Our MLE point estimator is then

$$\widehat{p}_1 = \frac{Y_1}{v}$$

with variance

$$\mathrm{Var}(\widehat{p}_1) = \frac{p_1(1 - p_1)}{v}.$$

To arrive at our AVC estimator let

$$p_1(\widehat{\mathbf{F}}) = \mathrm{Pr}\{X_{1i^*} > X_{ji^*}, \forall j \neq 1, \ i^* = 1, \ldots, v^k | \widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_k\}$$

where i^* represents the number of the pseudo-replication and $p_i(\widehat{\mathbf{F}})$ is the probability system i is the best when the data are distributed as $\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_k$. Under $\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_k$, each pseudo-replication occurs with an equal probability of $1/v^k$. We can then write

$$p_1(\widehat{\mathbf{F}}) = \frac{\text{(the number of pseudo-replications where population 1 wins)}}{v^k}$$

implying that the numerator is Z_1 from (3.3). Therefore, we can define our AVC estimator as

$$\bar{p}_1 = p_1(\widehat{\mathbf{F}}).$$

Using standard definitions for conditional expectation and variance (Casella and Berger 1990, p. 158) we know

$$
\begin{aligned}
\mathrm{Var}(\widehat{p}_1) &= \mathrm{Var}[E(\widehat{p}_1|\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_k)] + \mathrm{E}[\mathrm{Var}(\widehat{p}_1|\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_k)] \\
&= \mathrm{Var}[p_1(\widehat{\mathbf{F}})] + \mathrm{E}[\mathrm{Var}(\widehat{p}_1|\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_k)] \\
&= \mathrm{Var}[\bar{p}_1] + \mathrm{E}[\mathrm{Var}(\widehat{p}_1|\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_k)].
\end{aligned}
$$

Since $\mathrm{E}[\mathrm{Var}(\widehat{p}_1|\widehat{F}_1, \widehat{F}_2, \ldots, \widehat{F}_k)] \geq 0$, we have shown that

$$
\mathrm{Var}(\bar{p}_1) \leq \mathrm{Var}(\widehat{p}_1).
$$

In the following section we quantify the size of this reduction in the AVC estimator variance for specific cases.

3.4.2 Small-Sample Results

We restrict our attention to continuous distributions for the X_{ji}s, which eliminates the possibility of ties among the observations. We let $\pi_{[k]}$ be the best population and assume all the remaining populations, $\pi_{[1]}, \ldots, \pi_{[k-1]}$ are identically distributed. This setup gives us the LFC for BEM. We also consider all population distributions to belong to the same parametric family. For illustrative purposes, let X represent an observation from $\pi_{[k]}$ and let O represent an observation from any of the remaining inferior populations.

First, consider $X \sim \exp(\lambda)$ and $O \sim \exp(\mu)$ and let $\lambda < \mu$, where $\lambda > 0$ and $\mu > 0$ are exponential rates. This particular example was suggested by Goldsman (1995) in some early discussions concerning this research. For $k = 2$ and $v = 2$,

we have $p_{[2]} = \Pr\{X > O\} = \mu/(\lambda + \mu)$ and $p_{[1]} = \Pr\{X < O\} = \lambda/(\lambda + \mu)$. We arbitrarily let k be the best population so $p_k = p_{[k]}$. From (3.2) we have

$$\mathrm{Var}(\widehat{p}_2) = \frac{\lambda\mu}{2(\lambda + \mu)^2}.$$

To find $\mathrm{Var}(\bar{p}_2)$, we must first find the covariance terms. From (3.6) we have

$$
\begin{aligned}
\xi_{1,0} &= \Pr\{X_1 > O_1; X_1 > O_2\} - p_2^2 \\
&= \Pr\{X_1 > M\} - p_2^2 \\
&= \int_0^\infty 2\mu e^{-(\lambda+\mu)m}(1 - e^{-\mu m})\, dm - p_2^2 \\
&= \frac{2\mu^2}{(\lambda + 2\mu)(\lambda + \mu)} - \frac{\mu^2}{(\lambda + \mu)^2} \\
&= \frac{\lambda\mu^2}{(\lambda + 2\mu)(\lambda + \mu)^2}
\end{aligned}
$$

where $M \equiv \mathrm{maximum}(O_1, O_2)$. Then from (3.7)

$$
\begin{aligned}
\xi_{0,1} &= \Pr\{X_1 > O_1; X_2 > O_1\} - p_2^2 \\
&= \Pr\{N > O_1\} - p_2^2 \\
&= \int_0^\infty e^{-2\lambda n}(1 - e^{-\mu n})2\lambda\, dn - p_2^2 \\
&= \frac{\mu}{2\lambda + \mu} - \frac{\mu^2}{(\lambda + \mu)^2} \\
&= \frac{\lambda^2\mu}{(\lambda + 2\mu)(\lambda + \mu)^2}
\end{aligned}
$$

where $N \equiv \text{minimum}(X_1, X_2)$. Finally, from (3.8)

$$
\begin{aligned}
\xi_{1,1} &= \Pr\{X_1 > O_1; X_1 > O_1\} - p_2^2 \\
&= \frac{\mu}{\lambda + \mu} - \frac{\mu^2}{(\lambda + \mu)^2} \\
&= \frac{\lambda\mu}{(\lambda + \mu)^2}.
\end{aligned}
$$

Inserting these results into (3.9) we have

$$
\begin{aligned}
\text{Var}(\bar{p}_2) &= \frac{1}{4}\left[\xi_{0,0} + \xi_{0,1} + \xi_{1,0} + \xi_{1,1}\right] \\
&= \frac{3}{4}\frac{\lambda\mu(\lambda^2 + 3\lambda\mu + \mu^2)}{(2\lambda + \mu)(\lambda + 2\mu)(\lambda + \mu)^2}.
\end{aligned}
\tag{3.10}
$$

To illustrate the variance reduction achieved by the AVC estimator, we display the ratio of $\text{Var}(\bar{p}_2)$ to $\text{Var}(\hat{p}_2)$ plotted against $\theta = p_2/p_1 = \mu/\lambda$ in Figure 3.1. These results are for exponential populations with $k = 2$ and $v = 2$. A ratio less than 1 indicates a variance reduction with AVC. The reduction in variance is on the order of roughly 20% over a range of θ between 1 and 4. This covers most of the practical range of θ included in standard tables for BEM and used by experimentors. Clearly, as θ increases, both estimators approach 1 and the associated variances approach 0.

In order to demonstrate the weak dependence of the variance reduction on the distribution of the X_{ji}, consider the following example for $k = 2$ and $v = 2$ where X and O are from continuous uniform populations. Specifically $X \sim U(0, B)$ and $O \sim U(0, A)$, where $0 < A < B$. Then we have $p_{[2]} = \Pr\{X > O\} = (2B - A)/2B$ and $p_{[1]} = \Pr\{X < O\} = A/2B$. From (3.2) we have

$$
\text{Var}(\hat{p}_2) = \frac{1}{8}\frac{A(2B - A)}{B^2}.
$$
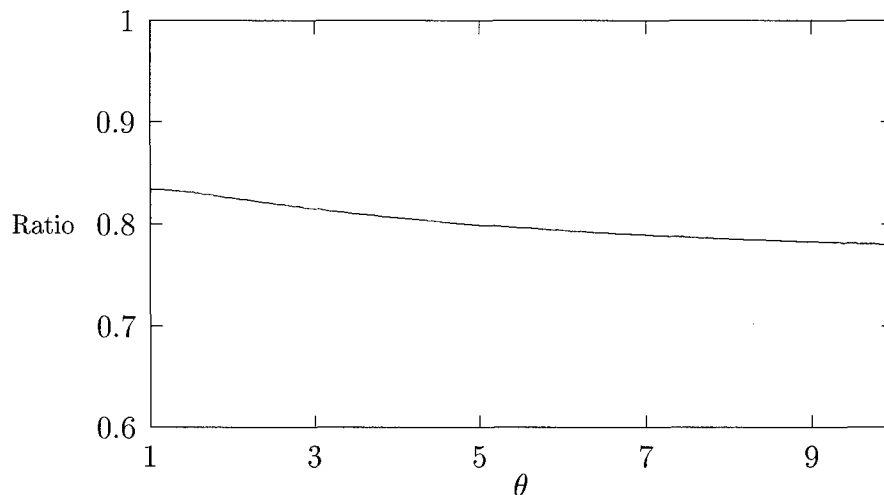
Figure 3.1: Ratio of $\mathrm{Var}(\bar{p}_2)/\mathrm{Var}(\hat{p}_2)$ for Exponential Populations with $k = 2$, $v = 2$

We can then find the corresponding variance of the AVC estimator in a similar fashion as we did for the exponential populations. From (3.9) we obtain

$$\mathrm{Var}(\bar{p}_2) = \frac{5A(2B - A)}{48B^2}.$$

In Figure 3.2 we show the reduction in the variance of the AVC estimator for our uniform populations again as a ratio of $\mathrm{Var}(\bar{p}_2)/\mathrm{Var}(\hat{p}_2)$. The MLE estimator variance is identical for both exponential and uniform distributions. We see a flatter curve in Figure 3.2 than in Figure 3.1. In Figure 3.3 we illustrate the difference between the exponential and uniform populations by plotting the ratio of $\mathrm{Var}(\bar{p}_2)$ for the uniform populations to $\mathrm{Var}(\bar{p}_2)$ for the exponential populations. It is interesting to notice that this difference is neglible over most of the practical range of θ.

64
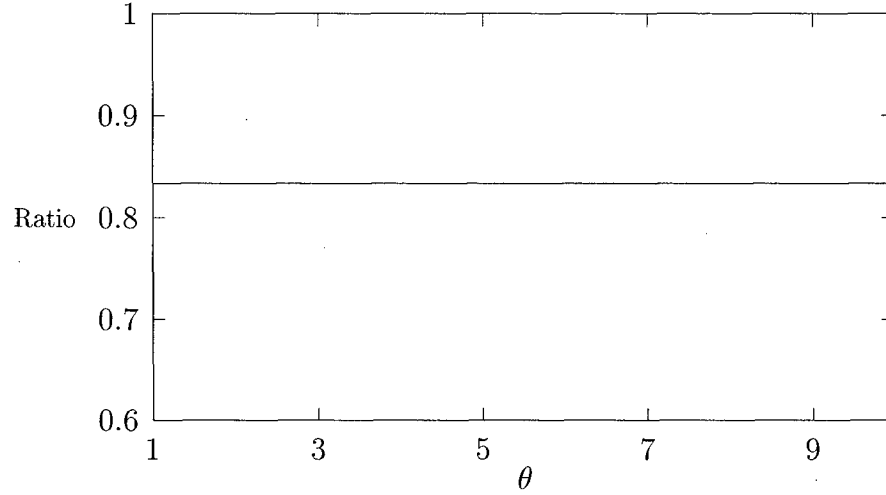
Figure 3.2: Ratio of $\mathrm{Var}(\bar{p}_2)/\mathrm{Var}(\hat{p}_2)$ for Uniform Populations with $k = 2$, $v = 2$

Once we have our covariance terms for a given k, we can find the exact variance for $\bar{p}_j$ for any v. For example with $k = 2$ using (3.9) we have

$$\mathrm{Var}(\bar{p}_j(v)) = \frac{1}{v^2}\left[(v - 1)\left(\xi_{0,1} + \xi_{1,0}\right) + \xi_{1,1}\right]. \tag{3.11}$$

As we increase k we add additional covariance terms to our variance calculations. For example at $k = 3$, we can modify (3.11) to get

$$\mathrm{Var}(\bar{p}_j(v)) = \frac{1}{v^3}\Big[(v - 1)^2\left(\xi_{1,0,0} + \xi_{0,1,0} + \xi_{0,0,1}\right) +$$
$$(v - 1)\left(\xi_{1,1,0} + \xi_{1,0,0} + \xi_{0,1,1}\right) + \xi_{1,1,1}\Big]$$

where $\xi_{c,d,e}$ are the analagous covariance terms for $k = 3$ populations, with $c, d, e \in \{0, 1\}$. These additional covariance terms become increasingly difficult to calculate

65
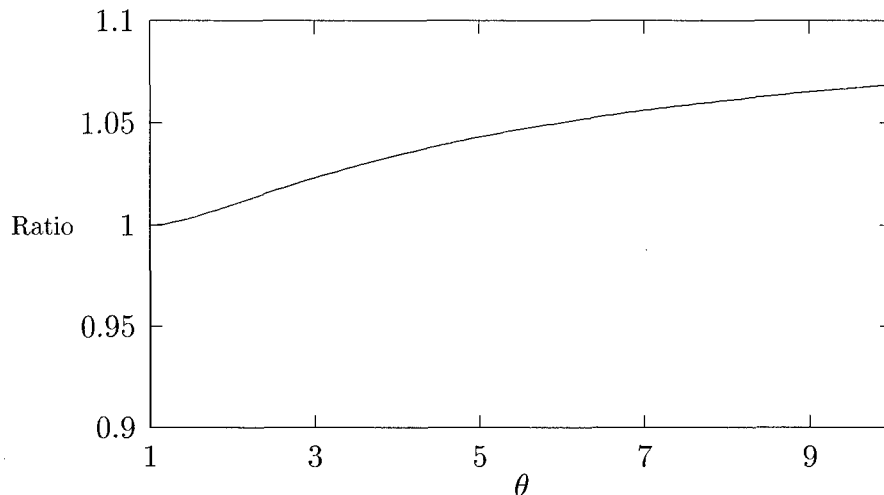
Figure 3.3: Ratio of Uniform $\text{Var}(\bar{p}_2)$ over Exponential $\text{Var}(\bar{p}_2)$ for $k = 2$, $v = 2$

for even moderately large k. This computational complexity, along with the weak distributional dependence of the AVC variance reduction, led us to consider an asymptotic approximation for the k-sample U-statistic variance.

3.5 Asymptotic Results

The results presented so far for small k and v show that the variance of the AVC estimator is weakly distribution dependent and tedious to compute. We derive asymptotic variance expressions for each of the estimators that are easy to estimate for all values of k.

3.5.1 Preliminaries

Using our previous notation we have

$$p_j = Pr\{X_{ji} > X_{\ell i}, \forall \ell \neq j\}.$$

Define

$$Y_j(v) \;=\; \text{number of wins for system } j \text{ under BEM for fixed } v$$

$$Z_j(v) \;=\; \text{number of wins for system } j \text{ under AVC for fixed } v$$

which gives us point estimators

$$\widehat{p}_j(v) \;=\; \frac{Y_j(v)}{v}$$

$$\bar{p}_j(v) \;=\; \frac{Z_j(v)}{v^k}.$$

Notice these are the same estimators we defined in (3.1) and (3.4) respectively, except that we have introduced a dependence on v. Our approach is based on the fact that standardized versions of $\widehat{p}$ and $\bar{p}$ are asymptotically multivariate normal (MVN).

3.5.2 MLEs and AVC Estimators

Consider the asymptotic behavior of the MLEs. It is well known that as $v \longrightarrow \infty$

$$\sqrt{v}(\widehat{p}_j - p_j) \overset{\mathcal{D}}{\Longrightarrow} \mathrm{N}\left(0, p_j(1 - p_j)\right). \tag{3.12}$$

Similarly, consider the asymptotic behavior of our AVC estimator as v goes to infinity. Following Randles and Wolfe (1979, pp. 105–106) let i be an integer such

that $1 \leq i \leq k$ and define

$$H_{i1}^{(a)} = h^{(a)}(X_{1\alpha_{11}}, \ldots, X_{k\alpha_{k1}})$$

and

$$H_{i2}^{(a)} = h^{(a)}(X_{1\beta_{11}}, \ldots, X_{k\beta_{k1}})$$

where

$$\begin{cases} \alpha_{j1} = \beta_{j1} & j = i \\ \alpha_{j1} \neq \beta_{j1} & j \neq i. \end{cases}$$

Then define the covariance terms

$$\begin{aligned} \xi_{0,\ldots,0,1,0,\ldots,0}^{(a,a)} &= \mathrm{Cov}\left[H_{i1}^{(a)}, H_{i2}^{(a)}\right] \\ &= \mathrm{E}\left[H_{i1}^{(a)} H_{i2}^{(a)}\right] - p_a^2 \end{aligned}$$

where the only 1 in the subscript of $\xi_{0,\ldots,0,1,0,\ldots,0}^{(a,a)}$ is in the ith position and $a \in \{1,\ldots,k\}$. With this notation a represents the population whose parameter we are estimating, and i represents the only population with a common observation in $H_{i1}^{(a)}$ and $H_{i2}^{(a)}$. Then using our kernel from (3.5) we have for $a = i$

$$\mathrm{E}[H_{i1}^{(a)} H_{i2}^{(a)}] = \mathrm{Pr}\left\{X_{a\alpha_a} > \max_{\ell \neq a}\{X_{\ell\alpha_\ell}, X_{\ell\beta_\ell}\}\right\}$$

and for $a \neq i$

$$\mathrm{E}[H_{i1}^{(a)} H_{i2}^{(a)}] = \mathrm{Pr}\left\{\begin{array}{l} X_{a\alpha_a} > X_{i\alpha_i}, X_{a\alpha_a} > \max_{\ell \neq a,i}\{X_{\ell\alpha_\ell}\}; \\ X_{a\beta_a} > X_{i\alpha_i}, X_{a\beta_a} > \max_{\ell \neq a,i}\{X_{\ell\beta_\ell}\} \end{array}\right\}.$$

This leaves us with just two covariance expressions for each population

$$\mathcal{L}^{(a)}(1) = \mathrm{Cov}[H_{a1}^{(a)}, H_{a2}^{(a)}] \tag{3.13}$$

and

$$\mathcal{L}^{(a)}(2) = \mathrm{Cov}[H_{i1}^{(a)}, H_{i2}^{(a)}], \quad a \neq i. \tag{3.14}$$

Let $N = kv$. We then state the following theorem due to Lehmann (1963). Some of the notation has been simplified for our context.

Theorem 3.1 *(Asymptotic Distribution of AVC Estimators)*

Let $U_a(X_{11}, \ldots, X_{1v}; \ldots; X_{k1}, \ldots, X_{kv}) = \bar{p}_a$ be a k-sample U-statistic for the parameter p_a of degree $(1, \ldots, 1)$. If $\lim_{v \to \infty}(v/N) = \lambda_i, 0 < \lambda_i < 1$, for $i = 1, \ldots, k$, and if $E[\{h^{(a)}(X_{11}, \ldots, X_{k1})\}^2] < \infty$, then $\sqrt{N}(\bar{p}_a - p_a)$ has a limiting normal distribution with mean 0 and variance

$$\sigma^2 = \sum_{i=1}^{k} \frac{\xi_{0,\ldots,1,0,\ldots,0}}{\lambda_i}, \tag{3.15}$$

provided $\sigma^2 > 0$.

Proof: See Lehmann (1963).

Using (3.13) and (3.14) we can simplify (3.15) as

$$\sigma^2 = k[\mathcal{L}^{(j)}(1) + (k-1)\mathcal{L}^{(j)}(2)]. \tag{3.16}$$

From Theorem 3.1 we have as $v \longrightarrow \infty$

$$\sqrt{N}(\bar{p}_j - p_j) \overset{\mathcal{D}}{\Longrightarrow} \mathrm{N}(0, \sigma^2) \quad j = 1, \ldots, k \tag{3.17}$$

where $N = kv$ and σ^2 is defined in (3.16). In comparing σ^2 with the asymptotic MLE variance in (3.12), we notice that the MLE expression is defined as $\sqrt{v} \longrightarrow \infty$ while the AVC expression is defined as $\sqrt{kv} \longrightarrow \infty$. To allow a fair comparison between these asymptotic variances, we multiply (3.12) through by $\sqrt{k}$ to obtain

$$\sqrt{k}(\sqrt{v}(\hat{p}_j - p_j)) \overset{\mathcal{D}}{\Longrightarrow} \mathrm{N}(0, k\, p_j(1 - p_j)) \tag{3.18}$$

as $\sqrt{kv} \longrightarrow \infty$. We now have comparable asymptotic variances for AVC and MLE from (3.17) and (3.18), respectively. In order to compare these asymptotic variances with the exact variance of $\bar{p}_k$ for any k we divide each expression by N to give us the following

$$\text{Var}_\mathcal{A}(\hat{p}_j) = \frac{1}{v} p_j (1 - p_j) \tag{3.19}$$

$$\text{Var}_\mathcal{A}(\bar{p}_j) \approx \frac{1}{v} [\mathcal{L}^{(j)}(1) + (k - 1)\mathcal{L}^{(j)}(2)]. \tag{3.20}$$

To compare these results we remove the dependence of the p_js on v by selecting a value for θ. Figure 3.4 plots $v \cdot \text{Var}_\mathcal{A}(\bar{p}_2)$, $v \cdot \text{Var}_\mathcal{A}(\hat{p}_2)$ and v times the exact variance found in (3.10) against v for $k = 2$ exponential populations with $\theta = 1.2$. This figure illustrates some very important facts about the relationship among the variances of our estimators. First, notice the significant reduction in the asymptotic approximation for our AVC variance over the MLE variance. Also notice how quickly the exact AVC variance approaches the AVC asymptotic approximation. At $v = 15$, the difference is only about 3%. This indicates that the asymptotic approximation for the AVC variance is quite good at relatively small values of v ($v \geq 15$) for $\theta = 1.2$. Calculations for larger values of θ at $v = 15$ show the difference between the exact and approximate AVC variance still about 3% for $\theta = 2.0$ and increasing to a difference of roughly 4% at $\theta = 3.0$. This shows that the $\text{Var}_\mathcal{A}(\bar{p}_2)$ provides a better approximation to $\text{Var}(\bar{p}_2)$ when the difference between systems is smaller.

With the favorable results presented above regarding the accuracy of our asymptotic approximation for AVC variance, we incorporate this approximate variance to calculate confidence intervals about our $\bar{p}_j$s. Because of the ease in
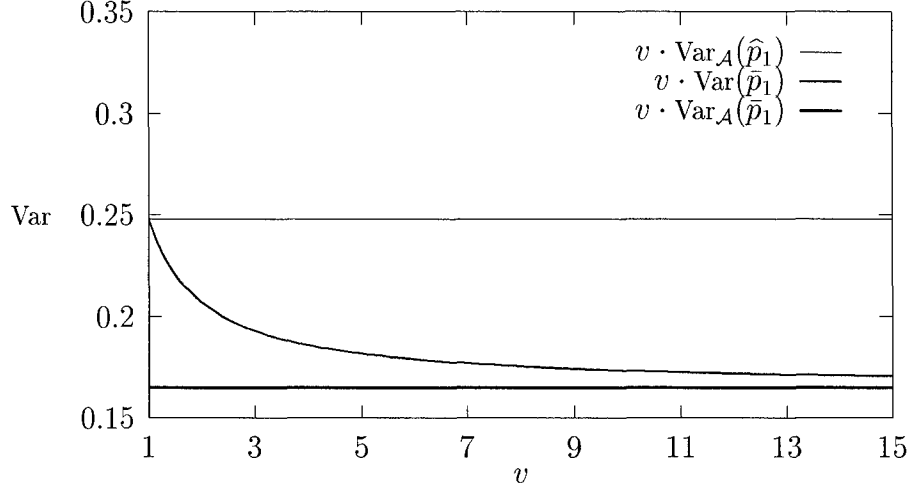
Figure 3.4: $v \cdot \text{Var}(\widehat{p}_2)$, $v \cdot \text{Var}(\bar{p}_2)$ and $v \cdot \text{Var}_{\mathcal{A}}(\bar{p}_2)$ for $k = 2$ Exponential Populations with $\theta = 1.2$

calculating the exact variance for the MLE estimators, we use this exact variance in constructing confidence intervals for the $\widehat{p}_j$s.

Suppose we have generated v independent observations from each of k populations. We compute $\widehat{p}_j$ and $\bar{p}_j$ $(j = 1, \ldots, k)$ using (3.1) and (3.4), respectively. We then estimate $\text{Var}_{\mathcal{A}}(\widehat{p}_j)$ and $\text{Var}_{\mathcal{A}}(\bar{p}_j)$ using (3.19) and (3.20), respectively. We perform M macro-replications of these kv observations and let $\widehat{p}_{ji}$, $\bar{p}_{ji}$, $\widehat{\text{Var}}_{\mathcal{A}}(\widehat{p}_j)_i$ and $\widehat{\text{Var}}_{\mathcal{A}}(\bar{p}_j)_i$ represent the point estimates and variances from the ith macro-replication. The sample variance is an unbiased estimator of the true variance so we calculate

$$\widehat{\text{Var}}_S(\bar{p}_j) \;\; = \;\; \frac{\sum_{i=1}^{M} \left(\bar{\bar{p}}_j - \bar{p}_{ji} \right)^2}{M-1}$$

where

$$\bar{\bar{p}}_j \;\; = \;\; \sum_{i=1}^{M} \frac{\bar{p}_{ji}}{M}.$$

We can also calculate average variances defined as

$$\overline{\text{Var}}_{\mathcal{A}}(\bar{p}_j) \;\; = \;\; \sum_{i=1}^{M} \frac{\widehat{\text{Var}}_{\mathcal{A}}(\bar{p}_j)_i}{M}$$

$$\overline{\text{Var}}_{\mathcal{A}}(\widehat{p}_j) \;\; = \;\; \sum_{i=1}^{M} \frac{\widehat{\text{Var}}_{\mathcal{A}}(\widehat{p}_j)_i}{M}.$$

We then estimate the bias of our AVC estimator variance as

$$\widehat{\text{Bias}} \left(\overline{\text{Var}}_{\mathcal{A}}(\bar{p}_j) \right) = \overline{\text{Var}}_{\mathcal{A}}(\bar{p}_j) - \widehat{\text{Var}}_S(\bar{p}_j).$$

Let $\mathcal{Z}_{\alpha/2}$ denote the $\alpha/2$ quantile of the standard normal distribution. In constructing our confidence intervals, we use the appropriate variance calculated in each macro-replication and then using normal approximations compute for the MLE

$$\widehat{p}_{ji} \pm \mathcal{Z}_{\alpha/2} \sqrt{\widehat{\text{Var}}_{\mathcal{A}}(\widehat{p}_j)_i}$$

and for AVC

$$\bar{p}_{ji} \pm \mathcal{Z}_{\alpha/2} \sqrt{\widehat{\text{Var}}_{\mathcal{A}}(\bar{p}_j)_i}.$$

We then count how many of the M intervals formed with each method capture p_j and compute the average confidence interval width.

We provide empirical results comparing the intervals between the two methods in the following section.

3.6 Empirical Study

For our empirical study we focus on the difference in the variance of the MLEs as compared to the variance of our AVC estimators. These comparisons take two different forms. First we simply examine the differences in the variances between the two methods. Since our AVC variance estimator is an approximation, we also obtain an unbiased estimator of the true AVC variance and compute an estimated bias for the AVC variance estimator. The variance reduction obtained is shown as $\widehat{\text{Var}}_{\mathcal{S}}(\bar{p}_k)/\overline{\text{Var}}_{\mathcal{A}}(\hat{p}_k)$. We then construct confidence intervals for the MLEs and AVC estimates and compare the percentage of coverage and average confidence interval width. All results are computed for $p_k = p_{[k]}$ using exponential population distributions as described in §3.4.2. Results are based on M=10,000 macro-replications for each of the v values listed.

Table 3.1 presents the variance results for each method for $k = 2$ to 5 populations and a number of different values of v. We start off with $v = 20$ to see how well our approximation works for small values of v. We find a significant negative bias in our AVC estimator variance at $v = 20$ for all k. At $v = 50$ this bias drops to between 6% and 7% for $k = 2$ and less than 10% for $k = 3$. At $k = 5$ the bias is stll more than 10%. Moving up to $v = 100$, the bias for both $k = 2$ and 3 drops below 7% and for $k = 5$ the bias drops below 10%. At $v = 200$ the bias effectively goes to zero for all k. These results indicate that we can substantially underestimate $\text{Var}(\bar{p}_k)$ by using $\widehat{\text{Var}}_{\mathcal{A}}(\bar{p}_k)$ for $v \leq 50$. The last column of Table 3.1 shows the ratio of $\widehat{\text{Var}}_{\mathcal{S}}(\bar{p}_k)/\overline{\text{Var}}_{\mathcal{A}}(\hat{p}_k)$ to illustrate the variance reduction with the

AVC estimator. Focusing on the results for $v = 200$ at $\theta^* = 1.2$, we see roughly a 33% reduction at $k = 2$, roughly a 42% reduction at $k = 3$, and roughly a 45% reduction at $k = 5$.

Looking at the 95% confidence interval results in Table 3.2, we see poor coverage and relatively large intervals for $v \leq 50$ for both MLE and AVC. This indicates that the normal approximation is not particularly good for either method at small values of v. At $v = 100$ both MLE and AVC coverage jump up to 94% to 95% in nearly all cases, with MLE coverage slightly better than the AVC coverage. However, at $v = 200$ the AVC coverage slightly exceeds the MLE coverage, with both methods achieving 95% coverage in almost all case. In all cases the average AVC interval width shows a 20-25% reduction over the average MLE interval width.

3.7 Conclusions

We have shown that our AVC estimator for multinomial success probabilities has a no larger variance than the MLE. Our results indicate a reduction in variance on the order of 33% for $k = 2$, increasing to roughly 45% for $k = 5$ populations can be expected. We have also shown that this variance reduction depends weakly on the distribution of the performance measures from the populations. An asymptotic approximation for our AVC estimator variance was found and this was used to compare to the MLE variance directly, and through the construction of confidence intervals. For small v we saw that our AVC estimator variance has a relatively large negative bias. However, for $v \geq 100$, this bias became insignificant. The confidence interval results show similar percent coverage with AVC and MLE for

k	θ^*	v	$\widehat{\mathrm{Var}}_{\mathcal{S}}(\bar{p}_k)$	$\overline{\mathrm{Var}}_{\mathcal{A}}(\bar{p}_k)$	$\overline{\mathrm{Var}}_{\mathcal{A}}(\widehat{p}_k)$	$\mathrm{Bias}(\widehat{\mathrm{Var}}_{\mathcal{A}}(\bar{p}_k))$	$\dfrac{\widehat{\mathrm{Var}}_{\mathcal{S}(\bar{p}_k)}}{\overline{\mathrm{Var}}_{\mathcal{A}(\widehat{p}_k)}}$
	1.2	20	.0086	.0074	.0118	−.0012	.7288
2	1.2	50	.0034	.0032	.0049	−.0002	.6939
	1.2	100	.0017	.0016	.0025	−.0001	.6800
	1.2	200	.0008	.0008	.0012	.0000	.6667
	2.0	20	.0076	.0065	.0105	−.0011	.7238
2	2.0	50	.0030	.0028	.0044	−.0002	.6818
	2.0	100	.0015	.0014	.0022	−.0001	.6818
	2.0	200	.0007	.0007	.0011	.0000	.6364
	1.2	20	.0075	.0061	.0111	−.0014	.6757
3	1.2	50	.0030	.0027	.0046	−.0003	.6522
	1.2	100	.0015	.0014	.0023	−.0001	.6522
	1.2	200	.0007	.0007	.0012	.0000	.5833
	2.0	20	.0085	.0069	.0119	−.0016	.7143
3	2.0	50	.0034	.0031	.0049	−.0003	.6939
	2.0	100	.0017	.0016	.0025	−.0001	.6800
	2.0	200	.0008	.0008	.0012	.0000	.6667
	1.2	20	.0053	.0038	.0084	−.0015	.6310
5	1.2	50	.0021	.0018	.0035	−.0003	.6000
	1.2	100	.0011	.0010	.0018	−.0001	.6111
	1.2	200	.0005	.0005	.0009	.0000	.5555
	2.0	20	.0073	.0053	.0105	−.0020	.6952
5	2.0	50	.0029	.0026	.0043	−.0003	.6744
	2.0	100	.0015	.0014	.0022	−.0001	.6818
	2.0	200	.0007	.0007	.0011	.0000	.6364

Table 3.1: AVC and MLE Variance for p_k

k	θ^*	v	AVC		MLE	
			Percentage Coverage	Average Width	Percentage Coverage	Average Width
	1.2	20	.92	.34	.92	.42
2	1.2	50	.94	.22	.93	.27
	1.2	100	.94	.16	.94	.19
	1.2	200	.95	.11	.94	.14
	2.0	20	.91	.31	.92	.40
2	2.0	50	.94	.21	.95	.26
	2.0	100	.94	.15	.96	.18
	2.0	200	.95	.10	.95	.13
	1.2	20	.90	.31	.94	.41
3	1.2	50	.93	.20	.94	.27
	1.2	100	.94	.15	.95	.19
	1.2	200	.95	.10	.94	.13
	2.0	20	.91	.32	.96	.43
3	2.0	50	.93	.22	.93	.27
	2.0	100	.94	.16	.94	.20
	2.0	200	.95	.11	.95	.14
	1.2	20	.87	.24	.94	.35
5	1.2	50	.92	.17	.95	.23
	1.2	100	.93	.12	.95	.16
	1.2	200	.94	.09	.94	.12
	2.0	20	.88	.28	.93	.40
5	2.0	50	.92	.20	.94	.26
	2.0	100	.94	.14	.95	.18
	2.0	200	.95	.10	.95	.13

Table 3.2: AVC and MLE 95% Confidence Intervals for p_k

$v \geq 100$ with the average AVC interval width 20–25% less than MLE. Our AVC estimators make more efficient use of the data already available to provide a more precise set of estimators for the multinomial success probabilities.

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Appendix A

PROOFS

Proof of Theorem 2.1:

From (2.9), as $v \to \infty$ we have $p_i(v) \to 1/k$ for all i. Then from Lemma 2.1

$$\begin{pmatrix} A_2(v) \\ \vdots \\ A_k(v) \end{pmatrix} \Longrightarrow$$

$$\text{MVN}\left[\begin{pmatrix} -\delta \\ \vdots \\ -\delta \end{pmatrix}, \begin{pmatrix} 1/k(1-1/k) & -1/k^2 & \cdots & -1/k^2 \\ -1/k^2 & 1/k(1-1/k) & \cdots & -1/k^2 \\ \vdots & \vdots & \ddots & \vdots \\ -1/k^2 & -1/k^2 & \cdots & 1/k(1-1/k) \end{pmatrix} \right].$$

Since $A_i(v) - A_1(v) = A_i(v) + \sum_{j=2}^{k} A_j(v)$ $(i \neq 1)$, we take the difference of MVN random variables and obtain

$$\begin{pmatrix} A_2(v) - A_1(v) \\ \vdots \\ A_k(v) - A_1(v) \end{pmatrix} \Longrightarrow$$

$$\text{MVN}\left[\begin{pmatrix} -k\delta \\ \vdots \\ -k\delta \end{pmatrix}, 2/k \begin{pmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 1 & \cdots & 1/2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & \cdots & 1 \end{pmatrix} \right]. \qquad (A.1)$$

Assuming population 1 is the best, in terms of $\overrightarrow{\text{PCS}}^{\text{bem}}$ we can state

$$
\begin{aligned}
\overrightarrow{\text{PCS}}^{\text{bem}} &= \lim_{v \longrightarrow \infty} \Pr\left\{Y_1(v) > Y_j(v), \ \forall j \neq 1\right\} \\
&= \lim_{v \longrightarrow \infty} \Pr\left\{\frac{Y_1(v) - v/k}{\sqrt{v}} > \frac{Y_j(v) - v/k}{\sqrt{v}}, \ \forall j \neq 1\right\} \\
&= \lim_{v \longrightarrow \infty} \Pr\{A_1(v) > A_j(v), \ \forall j \neq 1\} \\
&= \lim_{v \longrightarrow \infty} \Pr\{A_j(v) - A_1(v) < 0, \ \forall j \neq 1\} \\
&= \Pr\{W_j < 0, \ j = 2, 3, \ldots, k\} \qquad\qquad\qquad \text{(A.2)}
\end{aligned}
$$

where $(W_2, \ldots, W_k) \sim$ (A.1). If we add $k\delta$ to each W_j to obtain a random vector with a mean of zero, then from (A.2) we have

$$
\begin{aligned}
\overrightarrow{\text{PCS}}^{\text{bem}} &= \Pr\left\{\frac{W_j + k\delta}{\sqrt{2/k}} < \frac{k\delta}{\sqrt{2/k}}, \quad j = 2, \ldots, k\right\} \\
&= \Pr\left\{Q_j < \frac{k\delta}{\sqrt{2/k}}, \quad j = 2, \ldots, k\right\}
\end{aligned}
$$

where

$$
\begin{pmatrix} Q_2 \\ \vdots \\ Q_k \end{pmatrix} \sim \text{MVN}\left[\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 1 & \cdots & 1/2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & \cdots & 1 \end{pmatrix} \right].
$$

$\square$

Proof of Theorem 2.2:

Define $\mathbf{\Sigma}(N)$, the covariance matrix computed for $(F_1^{(v)}, \ldots, F_k^{(v)})$, as

$$
\mathbf{\Sigma}(N) = \text{Var}\begin{bmatrix} \bar{p}_1(N) \\ \vdots \\ \bar{p}_k(N) \end{bmatrix}.
$$

Let $\mathrm{E}_N[\bar{p}_j(N)] = p_j(N)$ where $\mathrm{E}_N[\bar{p}_j(N)]$ denotes the expected value at sample size N ($N = kv$). Lehmann (1963, pp. 964-965) shows that Lemma 2.2 holds even

if the distributions of the data depend upon the sample size provided

$$\Sigma(N) \longrightarrow \Sigma,$$

where Σ is nonsingular as $N \longrightarrow \infty$. Since $\sum_{j=1}^{k} \bar{p}_j = 1$, we work with $\bar{p}_2, \ldots, \bar{p}_k$ to be nonsingular. We assume

$$\mathrm{Var}\left(\begin{array}{c} \sqrt{N}(\bar{p}_2(N) - p_2(N)) \\ \vdots \\ \sqrt{N}(\bar{p}_k(N) - p_k(N)) \end{array}\right) \longrightarrow \Sigma$$

for any reasonable set of $F_i^{(v)}$. We define Σ as $\|\sigma^{(a,b)}\|$ in (2.12).

To proceed, we need to consider the covariance terms defined in Theorem 2.2.

$$\begin{aligned} \xi_i^{(a,b)} &= \mathrm{E}[H_{i1}^{(a)} H_{i2}^{(b)}] - \mathrm{E}[H_{i1}^{(a)}]\mathrm{E}[H_{i2}^{(b)}] \\ &= \mathrm{E}[H_{i1}^{(a)} H_{i2}^{(b)}] - 1/k^2 \end{aligned}$$

since

$$H_{i1}^{(a)} = \begin{cases} 1, & \text{if } X_{a\alpha_a} > X_{\ell\alpha_\ell} \ \forall \ell \neq a \\ 0, & \text{otherwise} \end{cases}$$

$$H_{i2}^{(b)} = \begin{cases} 1, & \text{if } X_{b\beta_b} > X_{\ell\beta_\ell} \ \forall \ell \neq b \\ 0, & \text{otherwise} \end{cases} .$$

Using our kernel from (2.11) we can express

$$\mathrm{E}[H_{a1}^{(a)} H_{a2}^{(a)}] = \mathrm{E}[\prod_{i=1,i\neq a}^{k} \phi(X_{a\alpha_a} - X_{i\alpha_i}) \times \prod_{\ell=1,\ell\neq a}^{k} \phi(X_{a\alpha_a} - X_{\ell\beta_\ell})].$$

We can then write this as a probability statement combining the two indices to come up with

$$\mathrm{E}[H_{a1}^{(a)} H_{a2}^{(a)}] = \mathrm{Pr}\left\{ X_{a\alpha_a} > \max_{\ell \neq a}\{X_{\ell\alpha_\ell}, X_{\ell\beta_\ell}\} \right\}.$$

80

This particular case is easy to illustrate and we extend this development for other expected value terms in less detail.

There are a number of different cases we need to consider for the covariance terms. In our notation $\xi_i^{(a,b)}$, a represents the population with the largest value in $H_i^{(a)}$; b represents the population with the largest value in $H_i^{(b)}$; and i represents the one population that has the same observation in both vectors (pseudo-replications). We can enumerate the different cases for the covariance terms based on the values of a, b and i. We have the following four cases.

1. $a = b = i$; $\xi_a^{(a,a)}$

2. $a = b \neq i$; $\xi_i^{(a,a)}$

3. $a \neq b \neq i$; $\xi_i^{(a,b)}$

4. $a \neq b$, $a = i$ or $b = i$; $\xi_a^{(a,b)}$ or $\xi_b^{(a,b)}$

Asymptotically $X_j \sim F$ $\forall j$ which allows us to construct distribution free expressions for $\xi_i^{(a,b)}$ since each X_{ij} has the same probability of being the largest value in a single vector. For case 1 we have

$$
\begin{aligned}
\xi_a^{(a,a)} &= \Pr\left\{ X_{a\alpha_a} > \max_{\ell \neq a}\{X_{\ell\alpha_\ell}, X_{\ell\beta_\ell}\} \right\} - \frac{1}{k^2} \\
&= \frac{1}{2k-1} - \frac{1}{k^2}.
\end{aligned}
\tag{A.3}
$$

This follows since we have $2(k-1) + 1$ independent and identically distributed random variables and we want the probability that a particular one is the largest. In our context this means we want the probability that a single observation from population a is a winner in two separate vectors containing no other common observations.

Case 2 is more difficult to approach. We have

$$\xi_i^{(a,a)} = \Pr\left\{ \begin{array}{l} X_{a\alpha_a} > X_{i\alpha_i}, X_{a\alpha_a} > \max_{\ell \neq a, i}\{X_{\ell\alpha_\ell}\}; \\ X_{a\beta_a} > X_{i\alpha_i}, X_{a\beta_a} > \max_{\ell \neq a, i}\{X_{\ell\beta_\ell}\} \end{array} \right\} - \frac{1}{k^2}$$

where the common observation in each of the two vectors is not the largest, and both observations from population a are the largest in their respective vectors. The ordering of the random variables from this pair of vectors must look like the following:

$$X_{a\alpha_a} > \{h \text{ of the } X_{\ell\alpha_\ell}\} > X_{a\beta_a} >$$

$$\{(k-2) - h \text{ of the } X_{\ell\alpha_\ell}, (k-2) \text{ of the } X_{\ell\beta_\ell}, \text{ and } X_{i\alpha_i}\}$$

or interchanging $X_{a\alpha_a}$ and $X_{a\beta_a}$

$$X_{a\beta_a} > \{h \text{ of the } X_{\ell\beta_\ell}\} > X_{a\alpha_a} >$$

$$\{(k-2) - h \text{ of the } X_{\ell\beta_\ell}, (k-2) \text{ of the } X_{\ell\alpha_\ell}, \text{ and } X_{i\alpha_i}\}$$

where $h = 0, 1, \ldots, k-2$. For each subset of size h there are

$$h!((k-2) - h + (k-2) + 1)!$$

equally likely orderings and there are

$$\binom{k-2}{h}$$

ways to select an h. With a total of $(2k-1)!$ possible orderings we then have

$$\xi_i^{(a,a)} = \frac{2 \sum_{h=0}^{k-2} \binom{k-2}{h} h! \left((k-2) - h + (k-2) + 1\right)!}{(2k-1)!} - \frac{1}{k^2}$$

$$= \frac{2}{k(2k-1)} - \frac{1}{k^2}. \tag{A.4}$$

We proceed in a similar fashion for case 3

$$
\begin{aligned}
\xi_i^{(a,b)} &= \Pr \left\{ \begin{array}{l} X_{a\alpha_a} > X_{i\alpha_i}, X_{a\alpha_a} > \max_{\ell \neq a,i}\{X_{\ell\alpha_\ell}\}; \\ X_{b\beta_b} > X_{i\alpha_i}, X_{b\beta_b} > \max_{\ell \neq b,i}\{X_{\ell\beta_\ell}\} \end{array} \right\} - \frac{1}{k^2} \\
&= \xi_i^{(a,a)} \\
&= \frac{2}{k(2k-1)} - \frac{1}{k^2}
\end{aligned}
\tag{A.5}
$$

since all random variables are identically distributed and there is no distinction

between $X_{b\beta_b}$ and $X_{a\beta_a}$.

For case 4 we have

$$
\xi_a^{(a,b)} = \Pr \left\{ X_{a\alpha_a} > \max_{\ell \neq a}\{X_{\ell\alpha_\ell}\}; X_{b\beta_b} > X_{a\alpha_a}, X_{b\beta_b} > \max_{\ell \neq a,b}\{X_{\ell\beta_\ell}\} \right\} - \frac{1}{k^2}.
$$

As we did for $\xi_a^{(a,a)}$ previously, we need to identify all possible arrangements of the
random variables from two vectors that meet the above conditions. The following
orderings work

$$
X_{b\beta_b} > \{h \text{ of the } X_{\ell\beta_\ell}\} > X_{a\alpha_a} > \{(k-2) - h \text{ of the } X_{\ell\beta_\ell}, \ (k-1) \text{ of the } X_{\ell\alpha_\ell}\}.
$$

Then proceeding as we did for $\xi_a^{(a,a)}$ we obtain for $a \neq b$

$$
\begin{aligned}
\xi_a^{(a,b)} &= \frac{\sum_{h=0}^{k-2} \binom{k-2}{h} h!\,((k-2) - h + (k-1))!}{(2k-1)!} - \frac{1}{k^2} \\
&= \frac{1}{k(2k-1)} - \frac{1}{k^2}.
\end{aligned}
\tag{A.6}
$$

We get identical results for $\xi_b^{(a,b)}$ with $a \neq b$.

Given covariance expressions from (A.3) and (A.4) we find the diagonal terms
of $\sigma^{(a,a)}$ from (2.12) as

$$
\begin{aligned}
\sigma^{(a,a)} &= k \sum_{i=1}^{k} \xi_i^{(a,a)} \\
&= k \left[\xi_a^{(a,a)} + (k-1)\xi_i^{(a,a)} \right] \\
&= k \left[\frac{1}{2k-1} - \frac{1}{k^2} + (k-1)\left(\frac{2}{k(2k-1)} - \frac{1}{k^2} \right) \right] \\
&= \frac{k-1}{2k-1}.
\end{aligned} \tag{A.7}
$$

With the covariance expressions from (A.5) and (A.6) we can find the off-diagonal terms of $\sigma^{(a,b)}$ from (2.12) as

$$
\begin{aligned}
\sigma^{(a,b)} &= k \left[\xi_a^{(a,b)} + \xi_b^{(a,b)} + \sum_{\ell \neq a,b} \xi_\ell^{(a,b)} \right] \\
&= k \left[2\left(\frac{1}{k(2k-1)} - \frac{1}{k^2} \right) + (k-2)\left(\frac{2}{k(2k-1)} - \frac{1}{k^2} \right) \right] \\
&= \frac{-1}{2k-1}.
\end{aligned} \tag{A.8}
$$

Combining the terms in (A.8) with those in (A.7) we let $\eta = 1/(k-1)$ and we have

$$
\Sigma = \left(\frac{k-1}{2k-1} \right) \begin{pmatrix} 1 & -\eta & \cdots & -\eta \\ -\eta & 1 & \cdots & -\eta \\ \vdots & \vdots & \ddots & \vdots \\ -\eta & -\eta & \cdots & 1 \end{pmatrix}. \tag{A.9}
$$

From Lehman (1963) we know that

$$
\begin{pmatrix} \sqrt{N}(\bar{p}_2(N) - (1/k - \delta/\sqrt{v})) \\ \vdots \\ \sqrt{N}(\bar{p}_k(N) - (1/k - \sigma/\sqrt{v})) \end{pmatrix} \implies \text{MVN} \left[\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \Sigma \right]. \tag{A.10}
$$

Asymptotically each $p_j = 1/k$ with our model, so each difference in (A.10) becomes $\sqrt{N}(\bar{p}_j(N) - 1/k)$. The resultant shift in the mean of (A.10) can be found as

$$
\begin{aligned}
\sqrt{N}(\bar{p}_j(N) - 1/k) &= \sqrt{N}(\bar{p}_j(N) - (1/k - \delta/\sqrt{v})) - \sqrt{N}\delta/\sqrt{v} \\
&= \sqrt{N}(\bar{p}_j(N) - (1/k) - \delta/\sqrt{v})) - \sqrt{k}\delta
\end{aligned}
$$

84

for $j = 2, \ldots, k$. Therefore, we have

$$\begin{pmatrix} \sqrt{N}(\bar{p}_2(N) - 1/k) \\ \vdots \\ \sqrt{N}(\bar{p}_k(N) - 1/k) \end{pmatrix} \implies \text{MVN} \left[\begin{pmatrix} -\sqrt{k}\delta \\ \vdots \\ -\sqrt{k}\delta \end{pmatrix}, \Sigma \right].$$

Assuming population 1 is the best, in terms of $\overrightarrow{\text{PCS}}^{\text{avc}}$ we have

$$\begin{aligned} \overrightarrow{\text{PCS}}^{\text{avc}} &= \lim_{N \longrightarrow \infty} \Pr\{\bar{p}_1(N) > \bar{p}_j(N), \, \forall \, j \neq 1\} \\ &= \lim_{N \longrightarrow \infty} \Pr\{\sqrt{N}(\bar{p}_1(N) - 1/k) > \sqrt{N}(\bar{p}_j(N) - 1/k), \, \forall \, j \neq 1\} \\ &= \lim_{N \longrightarrow \infty} \Pr\{\sqrt{N}(\bar{p}_j(N) - 1/k) - \sqrt{N}(\bar{p}_1(N) - 1/k) < 0, \, \forall \, j \neq 1\} \\ &= \Pr\{\bar{W}_j < 0, \, \forall j \neq 1\} \end{aligned}$$

where

$$\begin{pmatrix} \bar{W}_2 \\ \vdots \\ \bar{W}_k \end{pmatrix} \sim \text{MVN} \left[\begin{pmatrix} -k\sqrt{k}\delta \\ \vdots \\ -k\sqrt{k}\delta \end{pmatrix}, \Xi \right].$$

Using our variance and covariance terms from (A.9) the diagonal terms of Ξ are

$$\begin{aligned} \sigma^{(a,a)} + \sigma^{(b,b)} - 2\sigma^{(a,b)} &= \frac{k-1}{2k-1}\left(1 + 1 - 2\left(\frac{-1}{k-1}\right)\right) \\ &= \frac{2k}{2k-1} \end{aligned}$$

and the off-diagonal terms are

$$\begin{aligned} \sigma^{(a,b)} - \sigma^{(a,1)} - \sigma^{(1,b)} + \sigma^{1,1} &= \frac{k-1}{2k-1}\left(\frac{-1}{k-1} + \frac{1}{k-1} + \frac{1}{k-1} + 1\right) \\ &= \frac{k}{2k-1} \end{aligned}$$

Combining these terms we have

$$\Xi = \frac{2k}{2k-1} \begin{pmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 1 & \cdots & 1/2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & \cdots & 1 \end{pmatrix}.$$

85

Then

$$
\begin{aligned}
\overrightarrow{\mathrm{PCS}}^{\mathrm{avc}} &= \Pr\left\{\frac{\bar{W}_j + k\sqrt{k}\delta}{\sqrt{2k/(2k-1)}} < \frac{k\sqrt{k}\delta}{\sqrt{2k/(2k-1)}}, \quad j = 2,\ldots,k\right\} \\
&= \Pr\left\{Q_j < \frac{k\delta}{\sqrt{2/(2k-1)}}, \quad j = 2,\ldots,k\right\}
\end{aligned}
$$

where

$$
\begin{pmatrix} Q_2 \\ \vdots \\ Q_k \end{pmatrix} \sim \mathrm{MVN}\left[\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 1/2 & \cdots & 1/2 \\ 1/2 & 1 & \cdots & 1/2 \\ \vdots & \vdots & \ddots & \vdots \\ 1/2 & 1/2 & \cdots & 1 \end{pmatrix}\right].
$$

$\square$

Appendix B

TABLES OF RESULTS

The following tables are based on 100,000 macro-replications. The ratio column is $\text{PCS}^{\text{avc}}/\text{PCS}^{\text{bem}}$ and the s.e. columns are the respective standard errors. The population distributions used are discussed in §2.6.

v	$\mathrm{PCS^{bem}}$	$\mathrm{PCS^{avc}}$	Ratio	s.e. $\mathrm{PCS^{bem}}$	s.e. $\mathrm{PCS^{avc}}$
2	0.5463	0.5566	1.0188	0.0016	0.0016
4	0.5684	0.5849	1.0290	0.0016	0.0016
6	0.5853	0.6058	1.0350	0.0016	0.0015
8	0.5998	0.6248	1.0416	0.0015	0.0015
10	0.6106	0.6370	1.0432	0.0015	0.0015
12	0.6218	0.6502	1.0457	0.0015	0.0015
14	0.6317	0.6615	1.0471	0.0015	0.0015
16	0.6405	0.6722	1.0495	0.0015	0.0015
18	0.6475	0.6813	1.0523	0.0015	0.0015
20	0.6559	0.6901	1.0521	0.0015	0.0015
22	0.6641	0.6981	1.0512	0.0015	0.0015
24	0.6711	0.7057	1.0515	0.0015	0.0014
26	0.6770	0.7133	1.0536	0.0015	0.0014
28	0.6825	0.7215	1.0572	0.0015	0.0014
30	0.6893	0.7277	1.0558	0.0015	0.0014
32	0.6944	0.7331	1.0557	0.0015	0.0014
34	0.7004	0.7386	1.0546	0.0014	0.0014
36	0.7053	0.7454	1.0567	0.0014	0.0014
38	0.7110	0.7520	1.0577	0.0014	0.0014
40	0.7164	0.7574	1.0573	0.0014	0.0014
42	0.7210	0.7635	1.0590	0.0014	0.0013
44	0.7257	0.7694	1.0602	0.0014	0.0013
46	0.7299	0.7727	1.0586	0.0014	0.0013
48	0.7342	0.7781	1.0598	0.0014	0.0013
50	0.7378	0.7832	1.0615	0.0014	0.0013

Table B.1: Simulation Results for Exponential Populations with $\theta = 1.2$ and $k = 2$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.3763	0.3900	1.0365	0.0015	0.0015
4	0.3978	0.4126	1.0371	0.0015	0.0016
6	0.4127	0.4317	1.0462	0.0016	0.0016
8	0.4242	0.4501	1.0610	0.0016	0.0016
10	0.4358	0.4643	1.0656	0.0016	0.0016
12	0.4454	0.4747	1.0658	0.0016	0.0016
14	0.4546	0.4871	1.0714	0.0016	0.0016
16	0.4639	0.4999	1.0775	0.0016	0.0016
18	0.4726	0.5106	1.0805	0.0016	0.0016
20	0.4799	0.5203	1.0841	0.0016	0.0016
22	0.4880	0.5299	1.0859	0.0016	0.0016
24	0.4932	0.5372	1.0892	0.0016	0.0016
26	0.4988	0.5454	1.0934	0.0016	0.0016
28	0.5057	0.5540	1.0954	0.0016	0.0016
30	0.5117	0.5613	1.0970	0.0016	0.0016
32	0.5169	0.5698	1.1024	0.0016	0.0016
34	0.5230	0.5747	1.0989	0.0016	0.0016
36	0.5279	0.5801	1.0987	0.0016	0.0016
38	0.5347	0.5881	1.0997	0.0016	0.0016
40	0.5398	0.5950	1.1022	0.0016	0.0016
42	0.5446	0.6021	1.1056	0.0016	0.0015
44	0.5487	0.6079	1.1080	0.0016	0.0015
46	0.5542	0.6129	1.1059	0.0016	0.0015
48	0.5583	0.6172	1.1056	0.0016	0.0015
50	0.5625	0.6247	1.1107	0.0016	0.0015

Table B.2: Simulation Results for Exponential Populations with $\theta = 1.2$ and $k = 3$

v	PCS^{bem}	PCS^{avc}	Ratio	s.e. PCS^{bem}	s.e. PCS^{avc}
2	0.2853	0.2983	1.0456	0.0014	0.0014
4	0.3032	0.3173	1.0465	0.0015	0.0015
6	0.3150	0.3352	1.0643	0.0015	0.0015
8	0.3258	0.3501	1.0745	0.0015	0.0015
10	0.3335	0.3614	1.0834	0.0015	0.0015
12	0.3427	0.3712	1.0831	0.0015	0.0015
14	0.3492	0.3818	1.0935	0.0015	0.0015
16	0.3573	0.3922	1.0978	0.0015	0.0015
18	0.3640	0.4010	1.1016	0.0015	0.0016
20	0.3716	0.4103	1.1041	0.0015	0.0016
22	0.3795	0.4203	1.1075	0.0015	0.0016
24	0.3839	0.4266	1.1113	0.0015	0.0016
26	0.3893	0.4350	1.1173	0.0015	0.0016
28	0.3949	0.4431	1.1220	0.0015	0.0016
30	0.4000	0.4513	1.1282	0.0015	0.0016
32	0.4050	0.4569	1.1281	0.0016	0.0016
34	0.4093	0.4637	1.1329	0.0016	0.0016
36	0.4147	0.4680	1.1286	0.0016	0.0016
38	0.4184	0.4750	1.1352	0.0016	0.0016
40	0.4245	0.4820	1.1354	0.0016	0.0016
42	0.4287	0.4886	1.1398	0.0016	0.0016
44	0.4327	0.4936	1.1408	0.0016	0.0016
46	0.4374	0.4993	1.1417	0.0016	0.0016
48	0.4421	0.5037	1.1393	0.0016	0.0016
50	0.4465	0.5100	1.1423	0.0016	0.0016

Table B.3: Simulation Results for Exponential Populations with $\theta = 1.2$ and $k = 4$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.2299	0.2413	1.0495	0.0013	0.0014
4	0.2443	0.2581	1.0566	0.0014	0.0014
6	0.2543	0.2709	1.0656	0.0014	0.0014
8	0.2637	0.2849	1.0806	0.0014	0.0014
10	0.2702	0.2952	1.0925	0.0014	0.0014
12	0.2786	0.3047	1.0938	0.0014	0.0015
14	0.2840	0.3122	1.0989	0.0014	0.0015
16	0.2898	0.3217	1.1098	0.0014	0.0015
18	0.2955	0.3289	1.1128	0.0014	0.0015
20	0.3029	0.3371	1.1129	0.0015	0.0015
22	0.3079	0.3470	1.1268	0.0015	0.0015
24	0.3130	0.3524	1.1261	0.0015	0.0015
26	0.3172	0.3596	1.1337	0.0015	0.0015
28	0.3224	0.3669	1.1382	0.0015	0.0015
30	0.3267	0.3736	1.1435	0.0015	0.0015
32	0.3305	0.3787	1.1458	0.0015	0.0015
34	0.3345	0.3854	1.1520	0.0015	0.0015
36	0.3390	0.3901	1.1507	0.0015	0.0015
38	0.3431	0.3955	1.1527	0.0015	0.0015
40	0.3477	0.4018	1.1558	0.0015	0.0016
42	0.3504	0.4074	1.1625	0.0015	0.0016
44	0.3555	0.4128	1.1612	0.0015	0.0016
46	0.3593	0.4180	1.1634	0.0015	0.0016
48	0.3632	0.4220	1.1619	0.0015	0.0016
50	0.3674	0.4267	1.1614	0.0015	0.0016

Table B.4: Simulation Results for Exponential Populations with $\theta = 1.2$ and $k = 5$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.1171	0.1216	1.0390	0.0010	0.0010
4	0.1222	0.1302	1.0660	0.0010	0.0011
6	0.1292	0.1370	1.0605	0.0011	0.0011
8	0.1338	0.1453	1.0859	0.0011	0.0011
10	0.1364	0.1503	1.1014	0.0011	0.0011
12	0.1403	0.1544	1.1008	0.0011	0.0011
14	0.1429	0.1595	1.1165	0.0011	0.0012
16	0.1462	0.1621	1.1091	0.0011	0.0012
18	0.1487	0.1664	1.1194	0.0011	0.0012
20	0.1518	0.1700	1.1196	0.0011	0.0012
22	0.1549	0.1747	1.1274	0.0011	0.0012
24	0.1571	0.1783	1.1348	0.0012	0.0012
26	0.1594	0.1819	1.1413	0.0012	0.0012
28	0.1614	0.1859	1.1520	0.0012	0.0012
30	0.1638	0.1901	1.1606	0.0012	0.0012
32	0.1666	0.1937	1.1626	0.0012	0.0012
34	0.1679	0.1960	1.1674	0.0012	0.0013
36	0.1708	0.1987	1.1638	0.0012	0.0013
38	0.1735	0.2016	1.1621	0.0012	0.0013
40	0.1759	0.2052	1.1665	0.0012	0.0013
42	0.1778	0.2084	1.1718	0.0012	0.0013
44	0.1800	0.2111	1.1732	0.0012	0.0013
46	0.1829	0.2137	1.1680	0.0012	0.0013
48	0.1846	0.2165	1.1730	0.0012	0.0013
50	0.1866	0.2190	1.1737	0.0012	0.0013

Table B.5: Simulation Results for Exponential Populations with $\theta = 1.2$ and $k = 10$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.6675	0.7003	1.0491	0.0015	0.0014
4	0.7408	0.7901	1.0665	0.0014	0.0013
6	0.7903	0.8428	1.0665	0.0013	0.0012
8	0.8270	0.8772	1.0606	0.0012	0.0010
10	0.8546	0.9040	1.0578	0.0011	0.0009
12	0.8779	0.9243	1.0528	0.0010	0.0008
14	0.8964	0.9387	1.0472	0.0010	0.0008
16	0.9117	0.9516	1.0437	0.0009	0.0007
18	0.9239	0.9604	1.0395	0.0008	0.0006
20	0.9347	0.9677	1.0353	0.0008	0.0006
22	0.9439	0.9739	1.0318	0.0007	0.0005
24	0.9522	0.9789	1.0281	0.0007	0.0005
26	0.9577	0.9826	1.0260	0.0006	0.0004
28	0.9635	0.9855	1.0228	0.0006	0.0004
30	0.9680	0.9882	1.0208	0.0006	0.0003
32	0.9727	0.9906	1.0184	0.0005	0.0003
34	0.9762	0.9923	1.0165	0.0005	0.0003
36	0.9792	0.9930	1.0141	0.0005	0.0003
38	0.9816	0.9943	1.0130	0.0004	0.0002
40	0.9837	0.9953	1.0118	0.0004	0.0002
42	0.9856	0.9959	1.0104	0.0004	0.0002
44	0.9875	0.9963	1.0089	0.0004	0.0002
46	0.9891	0.9973	1.0083	0.0003	0.0002
48	0.9905	0.9976	1.0071	0.0003	0.0002
50	0.9914	0.9978	1.0064	0.0003	0.0001

Table B.6: Simulation Results for Exponential Populations with $\theta = 2.0$ and $k = 2$

v	PCS^{bem}	PCS^{avc}	Ratio	s.e. PCS^{bem}	s.e. PCS^{avc}
2	0.5005	0.5560	1.1109	0.0016	0.0016
4	0.5936	0.6481	1.0917	0.0016	0.0015
6	0.6460	0.7133	1.1043	0.0015	0.0014
8	0.6868	0.7635	1.1116	0.0015	0.0013
10	0.7266	0.8022	1.1041	0.0014	0.0013
12	0.7579	0.8345	1.1011	0.0014	0.0012
14	0.7827	0.8600	1.0988	0.0013	0.0011
16	0.8082	0.8823	1.0917	0.0012	0.0010
18	0.8280	0.9007	1.0878	0.0012	0.0009
20	0.8457	0.9157	1.0827	0.0011	0.0009
22	0.8638	0.9289	1.0753	0.0011	0.0008
24	0.8756	0.9376	1.0707	0.0010	0.0008
26	0.8879	0.9468	1.0663	0.0010	0.0007
28	0.9003	0.9554	1.0612	0.0009	0.0007
30	0.9097	0.9609	1.0562	0.0009	0.0006
32	0.9185	0.9666	1.0525	0.0009	0.0006
34	0.9260	0.9718	1.0495	0.0008	0.0005
36	0.9334	0.9757	1.0453	0.0008	0.0005
38	0.9392	0.9790	1.0424	0.0008	0.0005
40	0.9444	0.9822	1.0400	0.0007	0.0004
42	0.9507	0.9842	1.0352	0.0007	0.0004
44	0.9547	0.9866	1.0334	0.0007	0.0004
46	0.9590	0.9887	1.0310	0.0006	0.0003
48	0.9629	0.9901	1.0282	0.0006	0.0003
50	0.9670	0.9908	1.0247	0.0006	0.0003

Table B.7: Simulation Results for Exponential Populations with $\theta = 2.0$ and $k = 3$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.4002	0.4554	1.1377	0.0015	0.0016
4	0.4867	0.5415	1.1124	0.0016	0.0016
6	0.5319	0.6086	1.1442	0.0016	0.0015
8	0.5788	0.6632	1.1458	0.0016	0.0015
10	0.6154	0.7072	1.1492	0.0015	0.0014
12	0.6516	0.7458	1.1447	0.0015	0.0014
14	0.6786	0.7782	1.1469	0.0015	0.0013
16	0.7072	0.8069	1.1410	0.0014	0.0012
18	0.7313	0.8305	1.1356	0.0014	0.0012
20	0.7544	0.8522	1.1296	0.0014	0.0011
22	0.7758	0.8714	1.1233	0.0013	0.0011
24	0.7916	0.8845	1.1174	0.0013	0.0010
26	0.8090	0.8982	1.1102	0.0012	0.0010
28	0.8227	0.9105	1.1068	0.0012	0.0009
30	0.8366	0.9202	1.0999	0.0012	0.0009
32	0.8498	0.9303	1.0947	0.0011	0.0008
34	0.8604	0.9381	1.0904	0.0011	0.0008
36	0.8713	0.9454	1.0851	0.0011	0.0007
38	0.8805	0.9510	1.0801	0.0010	0.0007
40	0.8890	0.9573	1.0769	0.0010	0.0006
42	0.8984	0.9621	1.0709	0.0010	0.0006
44	0.9065	0.9667	1.0665	0.0009	0.0006
46	0.9125	0.9706	1.0637	0.0009	0.0005
48	0.9188	0.9729	1.0588	0.0009	0.0005
50	0.9249	0.9766	1.0559	0.0008	0.0005

Table B.8: Simulation Results for Exponential Populations with $\theta = 2.0$ and $k = 4$

v	PCS$^{\mathrm{bem}}$	PCS$^{\mathrm{avc}}$	Ratio	s.e. PCS$^{\mathrm{bem}}$	s.e. PCS$^{\mathrm{avc}}$
2	0.3333	0.3840	1.1521	0.0015	0.0015
4	0.4069	0.4634	1.1388	0.0016	0.0016
6	0.4509	0.5271	1.1689	0.0016	0.0016
8	0.4941	0.5816	1.1771	0.0016	0.0016
10	0.5307	0.6256	1.1789	0.0016	0.0015
12	0.5635	0.6674	1.1843	0.0016	0.0015
14	0.5931	0.7017	1.1831	0.0016	0.0014
16	0.6210	0.7338	1.1817	0.0015	0.0014
18	0.6474	0.7610	1.1754	0.0015	0.0013
20	0.6708	0.7857	1.1713	0.0015	0.0013
22	0.6923	0.8085	1.1680	0.0015	0.0012
24	0.7113	0.8273	1.1631	0.0014	0.0012
26	0.7300	0.8431	1.1548	0.0014	0.0012
28	0.7472	0.8592	1.1498	0.0014	0.0011
30	0.7633	0.8720	1.1424	0.0013	0.0011
32	0.7785	0.8847	1.1364	0.0013	0.0010
34	0.7905	0.8952	1.1325	0.0013	0.0010
36	0.8045	0.9054	1.1254	0.0013	0.0009
38	0.8158	0.9148	1.1214	0.0012	0.0009
40	0.8274	0.9230	1.1156	0.0012	0.0008
42	0.8379	0.9314	1.1116	0.0012	0.0008
44	0.8483	0.9381	1.1059	0.0011	0.0008
46	0.8562	0.9441	1.1027	0.0011	0.0007
48	0.8652	0.9479	1.0956	0.0011	0.0007
50	0.8732	0.9537	1.0922	0.0011	0.0007

Table B.9: Simulation Results for Exponential Populations with $\theta = 2.0$ and $k = 5$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.1820	0.2106	1.1572	0.0012	0.0013
4	0.2141	0.2596	1.2126	0.0013	0.0014
6	0.2497	0.3011	1.2058	0.0014	0.0015
8	0.2737	0.3390	1.2386	0.0014	0.0015
10	0.2974	0.3747	1.2599	0.0014	0.0015
12	0.3225	0.4067	1.2610	0.0015	0.0016
14	0.3439	0.4353	1.2656	0.0015	0.0016
16	0.3640	0.4628	1.2716	0.0015	0.0016
18	0.3835	0.4891	1.2754	0.0015	0.0016
20	0.4029	0.5147	1.2774	0.0016	0.0016
22	0.4212	0.5393	1.2805	0.0016	0.0016
24	0.4383	0.5610	1.2801	0.0016	0.0016
26	0.4548	0.5825	1.2808	0.0016	0.0016
28	0.4713	0.6022	1.2777	0.0016	0.0015
30	0.4864	0.6217	1.2783	0.0016	0.0015
32	0.5008	0.6407	1.2793	0.0016	0.0015
34	0.5154	0.6587	1.2780	0.0016	0.0015
36	0.5301	0.6752	1.2738	0.0016	0.0015
38	0.5437	0.6905	1.2699	0.0016	0.0015
40	0.5576	0.7055	1.2653	0.0016	0.0014
42	0.5693	0.7205	1.2656	0.0016	0.0014
44	0.5820	0.7340	1.2612	0.0016	0.0014
46	0.5947	0.7468	1.2557	0.0016	0.0014
48	0.6056	0.7581	1.2520	0.0015	0.0014
50	0.6171	0.7703	1.2482	0.0015	0.0013

Table B.10: Simulation Results for Exponential Populations with $\theta = 2.0$ and $k = 10$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.5468	0.5601	1.0244	0.0016	0.0016
4	0.5688	0.5881	1.0338	0.0016	0.0016
6	0.5849	0.6064	1.0368	0.0016	0.0015
8	0.5993	0.6221	1.0381	0.0016	0.0015
10	0.6122	0.6363	1.0395	0.0015	0.0015
12	0.6220	0.6490	1.0433	0.0015	0.0015
14	0.6322	0.6617	1.0466	0.0015	0.0015
16	0.6400	0.6704	1.0474	0.0015	0.0015
18	0.6479	0.6793	1.0484	0.0015	0.0015
20	0.6585	0.6900	1.0478	0.0015	0.0015
22	0.6647	0.6989	1.0514	0.0015	0.0015
24	0.6715	0.7073	1.0533	0.0015	0.0014
26	0.6779	0.7136	1.0527	0.0015	0.0014
28	0.6844	0.7225	1.0557	0.0015	0.0014
30	0.6913	0.7283	1.0535	0.0015	0.0014
32	0.6953	0.7349	1.0570	0.0015	0.0014
34	0.7017	0.7410	1.0561	0.0014	0.0014
36	0.7066	0.7485	1.0593	0.0014	0.0014
38	0.7115	0.7538	1.0595	0.0014	0.0014
40	0.7167	0.7572	1.0565	0.0014	0.0014
42	0.7214	0.7636	1.0584	0.0014	0.0013
44	0.7252	0.7679	1.0588	0.0014	0.0013
46	0.7305	0.7731	1.0582	0.0014	0.0013
48	0.7338	0.7787	1.0612	0.0014	0.0013
50	0.7376	0.7824	1.0607	0.0014	0.0013

Table B.11: Simulation Results for Uniform Populations with $\theta = 1.2$ and $k = 2$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.3759	0.4030	1.0720	0.0015	0.0016
4	0.3992	0.4197	1.0514	0.0015	0.0016
6	0.4133	0.4372	1.0580	0.0016	0.0016
8	0.4236	0.4534	1.0704	0.0016	0.0016
10	0.4351	0.4643	1.0671	0.0016	0.0016
12	0.4459	0.4795	1.0753	0.0016	0.0016
14	0.4536	0.4909	1.0820	0.0016	0.0016
16	0.4644	0.5007	1.0783	0.0016	0.0016
18	0.4717	0.5109	1.0830	0.0016	0.0016
20	0.4793	0.5215	1.0880	0.0016	0.0016
22	0.4877	0.5314	1.0896	0.0016	0.0016
24	0.4922	0.5406	1.0983	0.0016	0.0016
26	0.4977	0.5479	1.1008	0.0016	0.0016
28	0.5061	0.5570	1.1006	0.0016	0.0016
30	0.5127	0.5631	1.0984	0.0016	0.0016
32	0.5177	0.5709	1.1028	0.0016	0.0016
34	0.5229	0.5774	1.1042	0.0016	0.0016
36	0.5288	0.5857	1.1077	0.0016	0.0016
38	0.5338	0.5919	1.1088	0.0016	0.0016
40	0.5397	0.5975	1.1071	0.0016	0.0016
42	0.5443	0.6042	1.1102	0.0016	0.0015
44	0.5487	0.6097	1.1111	0.0016	0.0015
46	0.5533	0.6156	1.1126	0.0016	0.0015
48	0.5590	0.6205	1.1099	0.0016	0.0015
50	0.5619	0.6254	1.1131	0.0016	0.0015

Table B.12: Simulation Results for Uniform Populations with $\theta = 1.2$ and $k = 3$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.2866	0.3141	1.0959	0.0014	0.0015
4	0.3066	0.3237	1.0558	0.0015	0.0015
6	0.3150	0.3416	1.0846	0.0015	0.0015
8	0.3269	0.3527	1.0786	0.0015	0.0015
10	0.3342	0.3658	1.0945	0.0015	0.0015
12	0.3443	0.3793	1.1017	0.0015	0.0015
14	0.3506	0.3889	1.1092	0.0015	0.0015
16	0.3592	0.3963	1.1034	0.0015	0.0015
18	0.3645	0.4066	1.1154	0.0015	0.0016
20	0.3721	0.4155	1.1166	0.0015	0.0016
22	0.3792	0.4252	1.1212	0.0015	0.0016
24	0.3848	0.4320	1.1228	0.0015	0.0016
26	0.3891	0.4394	1.1295	0.0015	0.0016
28	0.3951	0.4462	1.1291	0.0015	0.0016
30	0.4005	0.4550	1.1362	0.0016	0.0016
32	0.4057	0.4604	1.1348	0.0016	0.0016
34	0.4100	0.4675	1.1402	0.0016	0.0016
36	0.4148	0.4743	1.1433	0.0016	0.0016
38	0.4197	0.4799	1.1435	0.0016	0.0016
40	0.4233	0.4859	1.1477	0.0016	0.0016
42	0.4286	0.4927	1.1494	0.0016	0.0016
44	0.4340	0.4964	1.1439	0.0016	0.0016
46	0.4370	0.5037	1.1527	0.0016	0.0016
48	0.4415	0.5075	1.1496	0.0016	0.0016
50	0.4464	0.5119	1.1467	0.0016	0.0016

Table B.13: Simulation Results for Uniform Populations with $\theta = 1.2$ and $k = 4$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.2307	0.2566	1.1122	0.0013	0.0014
4	0.2459	0.2660	1.0815	0.0014	0.0014
6	0.2544	0.2810	1.1045	0.0014	0.0014
8	0.2631	0.2914	1.1074	0.0014	0.0014
10	0.2703	0.3020	1.1172	0.0014	0.0015
12	0.2782	0.3130	1.1250	0.0014	0.0015
14	0.2843	0.3197	1.1247	0.0014	0.0015
16	0.2922	0.3278	1.1218	0.0014	0.0015
18	0.2959	0.3356	1.1342	0.0014	0.0015
20	0.3029	0.3437	1.1345	0.0015	0.0015
22	0.3075	0.3530	1.1480	0.0015	0.0015
24	0.3129	0.3597	1.1495	0.0015	0.0015
26	0.3154	0.3651	1.1578	0.0015	0.0015
28	0.3224	0.3714	1.1522	0.0015	0.0015
30	0.3259	0.3788	1.1624	0.0015	0.0015
32	0.3298	0.3833	1.1621	0.0015	0.0015
34	0.3346	0.3903	1.1666	0.0015	0.0015
36	0.3405	0.3960	1.1632	0.0015	0.0015
38	0.3436	0.4020	1.1699	0.0015	0.0016
40	0.3463	0.4071	1.1753	0.0015	0.0016
42	0.3519	0.4125	1.1720	0.0015	0.0016
44	0.3564	0.4161	1.1673	0.0015	0.0016
46	0.3597	0.4225	1.1746	0.0015	0.0016
48	0.3622	0.4276	1.1806	0.0015	0.0016
50	0.3670	0.4308	1.1739	0.0015	0.0016

Table B.14: Simulation Results for Uniform Populations with $\theta = 1.2$ and $k = 5$

v	PCS^{bem}	PCS^{avc}	Ratio	s.e. PCS^{bem}	s.e. PCS^{avc}
2	0.6676	0.7039	1.0543	0.0015	0.0014
4	0.7411	0.7897	1.0655	0.0014	0.0013
6	0.7896	0.8407	1.0648	0.0013	0.0012
8	0.8268	0.8750	1.0583	0.0012	0.0010
10	0.8543	0.9013	1.0550	0.0011	0.0009
12	0.8776	0.9214	1.0500	0.0010	0.0009
14	0.8957	0.9360	1.0450	0.0010	0.0008
16	0.9113	0.9483	1.0405	0.0009	0.0007
18	0.9238	0.9582	1.0372	0.0008	0.0006
20	0.9350	0.9658	1.0329	0.0008	0.0006
22	0.9441	0.9723	1.0299	0.0007	0.0005
24	0.9524	0.9769	1.0258	0.0007	0.0005
26	0.9578	0.9811	1.0243	0.0006	0.0004
28	0.9630	0.9836	1.0214	0.0006	0.0004
30	0.9680	0.9870	1.0197	0.0006	0.0004
32	0.9723	0.9893	1.0174	0.0005	0.0003
34	0.9758	0.9913	1.0159	0.0005	0.0003
36	0.9791	0.9922	1.0134	0.0005	0.0003
38	0.9817	0.9937	1.0122	0.0004	0.0003
40	0.9835	0.9945	1.0112	0.0004	0.0002
42	0.9851	0.9950	1.0101	0.0004	0.0002
44	0.9873	0.9959	1.0087	0.0004	0.0002
46	0.9888	0.9968	1.0081	0.0003	0.0002
48	0.9903	0.9971	1.0069	0.0003	0.0002
50	0.9910	0.9974	1.0065	0.0003	0.0002

Table B.15: Simulation Results for Uniform Populations with $\theta = 2.0$ and $k = 2$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.5003	0.5866	1.1725	0.0016	0.0016
4	0.5936	0.6521	1.0986	0.0016	0.0015
6	0.6460	0.7140	1.1052	0.0015	0.0014
8	0.6870	0.7626	1.1100	0.0015	0.0013
10	0.7266	0.8000	1.1009	0.0014	0.0013
12	0.7575	0.8316	1.0978	0.0014	0.0012
14	0.7826	0.8562	1.0940	0.0013	0.0011
16	0.8079	0.8787	1.0876	0.0012	0.0010
18	0.8274	0.8960	1.0830	0.0012	0.0010
20	0.8447	0.9104	1.0778	0.0011	0.0009
22	0.8618	0.9245	1.0728	0.0011	0.0008
24	0.8749	0.9340	1.0675	0.0010	0.0008
26	0.8874	0.9436	1.0633	0.0010	0.0007
28	0.8995	0.9515	1.0579	0.0010	0.0007
30	0.9080	0.9576	1.0546	0.0009	0.0006
32	0.9174	0.9633	1.0500	0.0009	0.0006
34	0.9256	0.9680	1.0459	0.0008	0.0006
36	0.9331	0.9725	1.0423	0.0008	0.0005
38	0.9388	0.9760	1.0396	0.0008	0.0005
40	0.9445	0.9795	1.0370	0.0007	0.0004
42	0.9499	0.9819	1.0336	0.0007	0.0004
44	0.9542	0.9846	1.0319	0.0007	0.0004
46	0.9582	0.9867	1.0298	0.0006	0.0004
48	0.9620	0.9880	1.0271	0.0006	0.0003
50	0.9660	0.9894	1.0243	0.0006	0.0003

Table B.16: Simulation Results for Uniform Populations with $\theta = 2.0$ and $k = 3$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.4009	0.4967	1.2391	0.0016	0.0016
4	0.4875	0.5469	1.1218	0.0016	0.0016
6	0.5323	0.6157	1.1566	0.0016	0.0015
8	0.5788	0.6658	1.1503	0.0016	0.0015
10	0.6154	0.7075	1.1496	0.0015	0.0014
12	0.6515	0.7446	1.1429	0.0015	0.0014
14	0.6799	0.7760	1.1414	0.0015	0.0013
16	0.7080	0.8028	1.1339	0.0014	0.0013
18	0.7314	0.8259	1.1291	0.0014	0.0012
20	0.7532	0.8466	1.1241	0.0014	0.0011
22	0.7755	0.8650	1.1154	0.0013	0.0011
24	0.7912	0.8790	1.1110	0.0013	0.0010
26	0.8083	0.8920	1.1035	0.0012	0.0010
28	0.8217	0.9054	1.1019	0.0012	0.0009
30	0.8359	0.9141	1.0935	0.0012	0.0009
32	0.8483	0.9242	1.0894	0.0011	0.0008
34	0.8604	0.9324	1.0836	0.0011	0.0008
36	0.8708	0.9396	1.0790	0.0011	0.0008
38	0.8800	0.9457	1.0747	0.0010	0.0007
40	0.8892	0.9526	1.0713	0.0010	0.0007
42	0.8984	0.9570	1.0652	0.0010	0.0006
44	0.9060	0.9620	1.0619	0.0009	0.0006
46	0.9123	0.9659	1.0588	0.0009	0.0006
48	0.9184	0.9683	1.0543	0.0009	0.0006
50	0.9243	0.9731	1.0527	0.0008	0.0005

Table B.17: Simulation Results for Uniform Populations with $\theta = 2.0$ and $k = 4$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.3337	0.4289	1.2852	0.0015	0.0016
4	0.4081	0.4722	1.1570	0.0016	0.0016
6	0.4513	0.5375	1.1911	0.0016	0.0016
8	0.4940	0.5882	1.1907	0.0016	0.0016
10	0.5293	0.6304	1.1910	0.0016	0.0015
12	0.5634	0.6688	1.1871	0.0016	0.0015
14	0.5924	0.7028	1.1863	0.0016	0.0014
16	0.6207	0.7317	1.1789	0.0015	0.0014
18	0.6457	0.7576	1.1732	0.0015	0.0014
20	0.6689	0.7801	1.1662	0.0015	0.0013
22	0.6921	0.8025	1.1596	0.0015	0.0013
24	0.7105	0.8196	1.1536	0.0014	0.0012
26	0.7294	0.8353	1.1452	0.0014	0.0012
28	0.7467	0.8519	1.1409	0.0014	0.0011
30	0.7610	0.8643	1.1358	0.0013	0.0011
32	0.7766	0.8774	1.1298	0.0013	0.0010
34	0.7905	0.8872	1.1223	0.0013	0.0010
36	0.8043	0.8979	1.1165	0.0013	0.0010
38	0.8158	0.9073	1.1122	0.0012	0.0009
40	0.8267	0.9150	1.1069	0.0012	0.0009
42	0.8368	0.9235	1.1036	0.0012	0.0008
44	0.8476	0.9305	1.0978	0.0011	0.0008
46	0.8564	0.9365	1.0935	0.0011	0.0008
48	0.8650	0.9412	1.0880	0.0011	0.0007
50	0.8732	0.9471	1.0847	0.0011	0.0007

Table B.18: Simulation Results for Uniform Populations with $\theta = 2.0$ and $k = 5$

v	PCS$^{\mathrm{bem}}$	PCS$^{\mathrm{avc}}$	Ratio	s.e. PCS$^{\mathrm{bem}}$	s.e. PCS$^{\mathrm{avc}}$
2	0.5438	0.5533	1.0175	0.0016	0.0016
4	0.5660	0.5762	1.0179	0.0016	0.0016
6	0.5826	0.5940	1.0196	0.0016	0.0016
8	0.5957	0.6098	1.0237	0.0016	0.0015
10	0.6082	0.6238	1.0257	0.0015	0.0015
12	0.6181	0.6351	1.0275	0.0015	0.0015
14	0.6285	0.6461	1.0281	0.0015	0.0015
16	0.6379	0.6565	1.0292	0.0015	0.0015
18	0.6458	0.6662	1.0316	0.0015	0.0015
20	0.6536	0.6732	1.0300	0.0015	0.0015
22	0.6615	0.6813	1.0299	0.0015	0.0015
24	0.6678	0.6888	1.0315	0.0015	0.0015
26	0.6745	0.6957	1.0314	0.0015	0.0015
28	0.6802	0.7030	1.0335	0.0015	0.0014
30	0.6862	0.7088	1.0329	0.0015	0.0014
32	0.6923	0.7145	1.0321	0.0015	0.0014
34	0.6980	0.7201	1.0316	0.0015	0.0014
36	0.7039	0.7258	1.0312	0.0014	0.0014
38	0.7092	0.7315	1.0313	0.0014	0.0014
40	0.7146	0.7374	1.0319	0.0014	0.0014
42	0.7201	0.7430	1.0318	0.0014	0.0014
44	0.7227	0.7481	1.0351	0.0014	0.0014
46	0.7276	0.7530	1.0349	0.0014	0.0014
48	0.7325	0.7572	1.0337	0.0014	0.0014
50	0.7371	0.7613	1.0328	0.0014	0.0013

Table B.19: Simulation Results for Gamma Populations with $\theta = 1.2$ and $k = 2$

v	PCS$^{\mathrm{bem}}$	PCS$^{\mathrm{avc}}$	Ratio	s.e. PCS$^{\mathrm{bem}}$	s.e. PCS$^{\mathrm{avc}}$
2	0.3742	0.3863	1.0323	0.0015	0.0015
4	0.3974	0.4071	1.0243	0.0015	0.0016
6	0.4112	0.4241	1.0313	0.0016	0.0016
8	0.4211	0.4392	1.0429	0.0016	0.0016
10	0.4340	0.4531	1.0441	0.0016	0.0016
12	0.4420	0.4648	1.0515	0.0016	0.0016
14	0.4506	0.4752	1.0547	0.0016	0.0016
16	0.4608	0.4853	1.0532	0.0016	0.0016
18	0.4689	0.4962	1.0584	0.0016	0.0016
20	0.4740	0.5048	1.0650	0.0016	0.0016
22	0.4832	0.5132	1.0621	0.0016	0.0016
24	0.4898	0.5222	1.0662	0.0016	0.0016
26	0.4958	0.5296	1.0682	0.0016	0.0016
28	0.5020	0.5369	1.0696	0.0016	0.0016
30	0.5076	0.5432	1.0700	0.0016	0.0016
32	0.5140	0.5500	1.0700	0.0016	0.0016
34	0.5199	0.5567	1.0708	0.0016	0.0016
36	0.5256	0.5627	1.0707	0.0016	0.0016
38	0.5309	0.5693	1.0723	0.0016	0.0016
40	0.5375	0.5758	1.0712	0.0016	0.0016
42	0.5435	0.5828	1.0722	0.0016	0.0016
44	0.5481	0.5882	1.0732	0.0016	0.0016
46	0.5538	0.5944	1.0732	0.0016	0.0016
48	0.5585	0.5995	1.0734	0.0016	0.0016
50	0.5628	0.6047	1.0744	0.0016	0.0015

Table B.20: Simulation Results for Gamma Populations with $\theta = 1.2$ and $k = 3$

v	PCS$^{\mathrm{bem}}$	PCS$^{\mathrm{avc}}$	Ratio	s.e. PCS$^{\mathrm{bem}}$	s.e. PCS$^{\mathrm{avc}}$
2	0.2863	0.2962	1.0346	0.0014	0.0014
4	0.3069	0.3180	1.0361	0.0015	0.0015
6	0.3153	0.3328	1.0558	0.0015	0.0015
8	0.3263	0.3445	1.0557	0.0015	0.0015
10	0.3355	0.3559	1.0607	0.0015	0.0015
12	0.3431	0.3687	1.0745	0.0015	0.0015
14	0.3505	0.3758	1.0722	0.0015	0.0015
16	0.3573	0.3847	1.0765	0.0015	0.0015
18	0.3637	0.3930	1.0806	0.0015	0.0015
20	0.3696	0.4008	1.0844	0.0015	0.0016
22	0.3755	0.4068	1.0832	0.0015	0.0016
24	0.3805	0.4145	1.0896	0.0015	0.0016
26	0.3862	0.4218	1.0922	0.0015	0.0016
28	0.3924	0.4283	1.0916	0.0015	0.0016
30	0.3977	0.4354	1.0947	0.0015	0.0016
32	0.4028	0.4419	1.0970	0.0016	0.0016
34	0.4081	0.4481	1.0980	0.0016	0.0016
36	0.4129	0.4538	1.0992	0.0016	0.0016
38	0.4170	0.4598	1.1027	0.0016	0.0016
40	0.4225	0.4654	1.1016	0.0016	0.0016
42	0.4265	0.4710	1.1045	0.0016	0.0016
44	0.4312	0.4773	1.1071	0.0016	0.0016
46	0.4357	0.4831	1.1089	0.0016	0.0016
48	0.4406	0.4888	1.1093	0.0016	0.0016
50	0.4445	0.4946	1.1126	0.0016	0.0016

Table B.21: Simulation Results for Gamma Populations with $\theta = 1.2$ and $k = 4$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.2315	0.2393	1.0337	0.0013	0.0013
4	0.2485	0.2589	1.0419	0.0014	0.0014
6	0.2560	0.2705	1.0565	0.0014	0.0014
8	0.2647	0.2801	1.0579	0.0014	0.0014
10	0.2735	0.2907	1.0630	0.0014	0.0014
12	0.2802	0.3025	1.0794	0.0014	0.0015
14	0.2850	0.3094	1.0857	0.0014	0.0015
16	0.2914	0.3166	1.0862	0.0014	0.0015
18	0.2973	0.3246	1.0919	0.0014	0.0015
20	0.3022	0.3308	1.0947	0.0015	0.0015
22	0.3072	0.3365	1.0953	0.0015	0.0015
24	0.3115	0.3431	1.1014	0.0015	0.0015
26	0.3168	0.3492	1.1024	0.0015	0.0015
28	0.3214	0.3548	1.1041	0.0015	0.0015
30	0.3268	0.3612	1.1053	0.0015	0.0015
32	0.3298	0.3667	1.1119	0.0015	0.0015
34	0.3337	0.3732	1.1183	0.0015	0.0015
36	0.3380	0.3796	1.1231	0.0015	0.0015
38	0.3424	0.3844	1.1227	0.0015	0.0015
40	0.3464	0.3893	1.1240	0.0015	0.0015
42	0.3516	0.3948	1.1228	0.0015	0.0015
44	0.3551	0.4001	1.1267	0.0015	0.0015
46	0.3586	0.4054	1.1307	0.0015	0.0016
48	0.3625	0.4103	1.1318	0.0015	0.0016
50	0.3665	0.4154	1.1334	0.0015	0.0016

Table B.22: Simulation Results for Gamma Populations with $\theta = 1.2$ and $k = 5$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.6672	0.7005	1.0499	0.0015	0.0014
4	0.7405	0.7782	1.0509	0.0014	0.0013
6	0.7895	0.8267	1.0472	0.0013	0.0012
8	0.8254	0.8608	1.0429	0.0012	0.0011
10	0.8553	0.8871	1.0372	0.0011	0.0010
12	0.8782	0.9077	1.0336	0.0010	0.0009
14	0.8959	0.9242	1.0315	0.0010	0.0008
16	0.9120	0.9377	1.0282	0.0009	0.0008
18	0.9248	0.9481	1.0252	0.0008	0.0007
20	0.9353	0.9569	1.0230	0.0008	0.0006
22	0.9446	0.9636	1.0202	0.0007	0.0006
24	0.9517	0.9700	1.0192	0.0007	0.0005
26	0.9578	0.9747	1.0177	0.0006	0.0005
28	0.9633	0.9785	1.0157	0.0006	0.0005
30	0.9681	0.9819	1.0142	0.0006	0.0004
32	0.9722	0.9846	1.0128	0.0005	0.0004
34	0.9760	0.9872	1.0115	0.0005	0.0004
36	0.9790	0.9890	1.0102	0.0005	0.0003
38	0.9815	0.9905	1.0091	0.0004	0.0003
40	0.9842	0.9917	1.0077	0.0004	0.0003
42	0.9861	0.9930	1.0070	0.0004	0.0003
44	0.9878	0.9940	1.0063	0.0003	0.0002
46	0.9893	0.9948	1.0056	0.0003	0.0002
48	0.9905	0.9955	1.0050	0.0003	0.0002
50	0.9917	0.9961	1.0044	0.0003	0.0002

Table B.23: Simulation Results for Gamma Populations with $\theta = 2.0$ and $k = 2$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.5013	0.5499	1.0969	0.0016	0.0016
4	0.5948	0.6367	1.0704	0.0016	0.0015
6	0.6463	0.6997	1.0826	0.0015	0.0014
8	0.6881	0.7473	1.0860	0.0015	0.0014
10	0.7282	0.7860	1.0793	0.0014	0.0013
12	0.7588	0.8170	1.0767	0.0014	0.0012
14	0.7843	0.8431	1.0750	0.0013	0.0012
16	0.8093	0.8649	1.0687	0.0012	0.0011
18	0.8295	0.8838	1.0656	0.0012	0.0010
20	0.8464	0.8999	1.0632	0.0011	0.0009
22	0.8628	0.9137	1.0591	0.0011	0.0009
24	0.8765	0.9256	1.0560	0.0010	0.0008
26	0.8879	0.9352	1.0532	0.0010	0.0008
28	0.8996	0.9434	1.0487	0.0010	0.0007
30	0.9093	0.9505	1.0453	0.0009	0.0007
32	0.9178	0.9567	1.0424	0.0009	0.0006
34	0.9259	0.9625	1.0395	0.0008	0.0006
36	0.9330	0.9675	1.0371	0.0008	0.0006
38	0.9392	0.9714	1.0342	0.0008	0.0005
40	0.9455	0.9752	1.0314	0.0007	0.0005
42	0.9504	0.9784	1.0294	0.0007	0.0005
44	0.9550	0.9807	1.0268	0.0007	0.0004
46	0.9588	0.9831	1.0254	0.0006	0.0004
48	0.9628	0.9850	1.0231	0.0006	0.0004
50	0.9661	0.9866	1.0211	0.0006	0.0004

Table B.24: Simulation Results for Gamma Populations with $\theta = 2.0$ and $k = 3$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.4008	0.4479	1.1174	0.0016	0.0016
4	0.4874	0.5372	1.1023	0.0016	0.0016
6	0.5324	0.5993	1.1256	0.0016	0.0016
8	0.5803	0.6506	1.1211	0.0016	0.0015
10	0.6175	0.6945	1.1247	0.0015	0.0015
12	0.6508	0.7311	1.1233	0.0015	0.0014
14	0.6800	0.7624	1.1211	0.0015	0.0013
16	0.7072	0.7901	1.1172	0.0014	0.0013
18	0.7317	0.8145	1.1132	0.0014	0.0012
20	0.7528	0.8356	1.1099	0.0014	0.0012
22	0.7729	0.8536	1.1045	0.0013	0.0011
24	0.7902	0.8702	1.1013	0.0013	0.0011
26	0.8065	0.8849	1.0973	0.0012	0.0010
28	0.8214	0.8973	1.0924	0.0012	0.0010
30	0.8354	0.9085	1.0876	0.0012	0.0009
32	0.8470	0.9183	1.0842	0.0011	0.0009
34	0.8589	0.9265	1.0787	0.0011	0.0008
36	0.8696	0.9349	1.0752	0.0011	0.0008
38	0.8786	0.9417	1.0718	0.0010	0.0007
40	0.8891	0.9476	1.0658	0.0010	0.0007
42	0.8971	0.9533	1.0626	0.0010	0.0007
44	0.9050	0.9582	1.0587	0.0009	0.0006
46	0.9121	0.9626	1.0554	0.0009	0.0006
48	0.9188	0.9660	1.0513	0.0009	0.0006
50	0.9246	0.9696	1.0486	0.0008	0.0005

Table B.25: Simulation Results for Gamma Populations with $\theta = 2.0$ and $k = 4$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.3334	0.3758	1.1273	0.0015	0.0015
4	0.4086	0.4594	1.1243	0.0016	0.0016
6	0.4519	0.5211	1.1530	0.0016	0.0016
8	0.4946	0.5728	1.1581	0.0016	0.0016
10	0.5330	0.6178	1.1591	0.0016	0.0015
12	0.5659	0.6562	1.1595	0.0016	0.0015
14	0.5945	0.6898	1.1605	0.0016	0.0015
16	0.6210	0.7189	1.1578	0.0015	0.0014
18	0.6477	0.7469	1.1533	0.0015	0.0014
20	0.6702	0.7708	1.1501	0.0015	0.0013
22	0.6913	0.7915	1.1449	0.0015	0.0013
24	0.7105	0.8124	1.1434	0.0014	0.0012
26	0.7294	0.8297	1.1375	0.0014	0.0012
28	0.7457	0.8460	1.1344	0.0014	0.0011
30	0.7617	0.8606	1.1298	0.0013	0.0011
32	0.7757	0.8726	1.1249	0.0013	0.0011
34	0.7900	0.8841	1.1190	0.0013	0.0010
36	0.8037	0.8946	1.1131	0.0013	0.0010
38	0.8151	0.9041	1.1092	0.0012	0.0009
40	0.8259	0.9130	1.1054	0.0012	0.0009
42	0.8370	0.9204	1.0995	0.0012	0.0009
44	0.8474	0.9275	1.0946	0.0011	0.0008
46	0.8567	0.9340	1.0902	0.0011	0.0008
48	0.8658	0.9399	1.0857	0.0011	0.0008
50	0.8733	0.9449	1.0820	0.0011	0.0007

Table B.26: Simulation Results for Gamma Populations with $\theta = 2.0$ and $k = 5$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.5453	0.5566	1.0208	0.0016	0.0016
4	0.5671	0.5920	1.0438	0.0016	0.0016
6	0.5838	0.6148	1.0532	0.0016	0.0015
8	0.5988	0.6357	1.0615	0.0016	0.0015
10	0.6100	0.6514	1.0679	0.0015	0.0015
12	0.6215	0.6661	1.0717	0.0015	0.0015
14	0.6309	0.6782	1.0749	0.0015	0.0015
16	0.6395	0.6901	1.0790	0.0015	0.0015
18	0.6479	0.7034	1.0857	0.0015	0.0014
20	0.6564	0.7132	1.0866	0.0015	0.0014
22	0.6631	0.7224	1.0894	0.0015	0.0014
24	0.6697	0.7320	1.0932	0.0015	0.0014
26	0.6764	0.7408	1.0952	0.0015	0.0014
28	0.6839	0.7503	1.0971	0.0015	0.0014
30	0.6896	0.7576	1.0986	0.0015	0.0014
32	0.6960	0.7642	1.0981	0.0015	0.0013
34	0.7018	0.7716	1.0996	0.0014	0.0013
36	0.7070	0.7785	1.1013	0.0014	0.0013
38	0.7118	0.7854	1.1034	0.0014	0.0013
40	0.7159	0.7907	1.1046	0.0014	0.0013
42	0.7209	0.7955	1.1036	0.0014	0.0013
44	0.7250	0.8016	1.1056	0.0014	0.0013
46	0.7293	0.8073	1.1070	0.0014	0.0012
48	0.7332	0.8122	1.1076	0.0014	0.0012
50	0.7387	0.8179	1.1072	0.0014	0.0012

Table B.27: Simulation Results for Bernoulli Populations with $\theta = 1.2$ and $k = 2$

v	PCS$^{\text{bem}}$	PCS$^{\text{avc}}$	Ratio	s.e. PCS$^{\text{bem}}$	s.e. PCS$^{\text{avc}}$
2	0.3744	0.3908	1.0438	0.0015	0.0015
4	0.3996	0.4244	1.0621	0.0015	0.0016
6	0.4126	0.4435	1.0748	0.0016	0.0016
8	0.4240	0.4647	1.0961	0.0016	0.0016
10	0.4357	0.4804	1.1026	0.0016	0.0016
12	0.4446	0.4948	1.1128	0.0016	0.0016
14	0.4523	0.5069	1.1208	0.0016	0.0016
16	0.4616	0.5204	1.1275	0.0016	0.0016
18	0.4690	0.5317	1.1337	0.0016	0.0016
20	0.4770	0.5442	1.1407	0.0016	0.0016
22	0.4841	0.5525	1.1413	0.0016	0.0016
24	0.4910	0.5656	1.1519	0.0016	0.0016
26	0.4984	0.5740	1.1518	0.0016	0.0016
28	0.5041	0.5845	1.1595	0.0016	0.0016
30	0.5106	0.5930	1.1613	0.0016	0.0016
32	0.5161	0.6017	1.1658	0.0016	0.0015
34	0.5223	0.6117	1.1711	0.0016	0.0015
36	0.5279	0.6196	1.1738	0.0016	0.0015
38	0.5338	0.6277	1.1759	0.0016	0.0015
40	0.5382	0.6337	1.1774	0.0016	0.0015
42	0.5436	0.6408	1.1788	0.0016	0.0015
44	0.5487	0.6483	1.1814	0.0016	0.0015
46	0.5536	0.6564	1.1857	0.0016	0.0015
48	0.5581	0.6623	1.1868	0.0016	0.0015
50	0.5635	0.6698	1.1885	0.0016	0.0015

Table B.28: Simulation Results for Bernoulli Populations with $\theta = 1.2$ and $k = 3$

Appendix C

PROGRAM LISTINGS

```
C****************************************************************
C****************************************************************
C****************************************************************
c
c  THIS IS PROGRAM FOR MULTINOMIAL SELECTION PROBLEM.  METHOD I
c  IS BECHHOFER'S METHOD (BEM).  METHOD II IS OUR NEW APPROACH
c  MAKING ALL VECTOR COMPARISONS (AVC).  BEST POPULATION IS
c  MODELED AS X1. SETUP TO USE THE FOLLOWING DISTRIBUTIONS:
c
c      EXPONENTIAL - SET LAMBDA=1.0, VARY MU TO MAINTAIN THETA
c      CONTINUOUS UNIFORM - SET A=1, VARY B TO MAINTAIN THETA
c      GAMMA - SET A1=A2=3.0, B2=1.0, VARY B1 TO MAINTAIN THETA
c
c  THIS IS REVISED VERSION SORTING AND THEN COMPARING EACH VALUE
c  AGAINST ALL VALUES FROM REMAINING POPULATIONS INSTEAD OF
c  ACTUALLY FORMING ALL V^K PSEUDO-REPLICATIONS.
c
c  SETUP FOR MAX OF 5 POPULATIONS AND 100 VECTORS.
c
C****************************************************************
C****************************************************************
c                   VARIABLE TABLE
c
c    NAME          DESCRIPTION
c
c    NREPS         NUMBER OF REPLICATIONS
c
```

```
c      NVECT          NUMBER OF INDEPENDENT VECTORS TO SIMULATE.
c                     SET UP FOR MAXIMUM OF 100.
c
c      NPOP           NUMBER OF POPULATIONS.  SET UP FOR MAXIMUM
c                     OF 5.
c
c      LAMBDA         EXPONENTIAL RATE FOR POPULATION 1, WHICH IS
c                     ALWAYS MODELED AS BEST SYSTEM.  SET TO 1.0.
c
c      MU             EXPONENTIAL RATE FOR POPULATIONS OTHER THAN
c                     1.  USE OF A SINGLE MU ASSUMES METHOD I
c                     l.f.c.  VARIED WITH INCREASING K TO MAINTAIN
c                     DESIRED THETA = P(X>Y)/(1-P(X>Y)/K-1).
c
c      A              RIGHT END POINT FOR CONTINUOUS UNIFORM DIST
c                     FOR POPULATION 1.  SET TO 1.0.
c
c      B              RIGHT END POINT FOR CONTINUOUS UNIFORM DIST
c                     FOR POPULATIONS OTHER THAN 1.  VARIED WITH
c                     INCREASING K TO MAINTAIN DESIRED THETA.
c
c      JX1(3)-JX5(3)  RANDOM NUMBER SEEDS FOR USE WITH L'ECUYER'S
c                     GENERATOR.  SEPARATE STREAM FOR EACH POP.
c
c      RUNF(JX)       CALL TO L'ECUYER'S GENERATOR.
c
c      PCS1(V)        PCS FOR METHOD I FOR V = THE NUMBER OF
c                     VECTORS, V=1,2,...,100.
c
c      PCS2(V)        PCS FOR METHOD II FOR V.
c
c      SE1(V)         STANDARD ERROR FOR PCS1(V).
c
c      SE2(V)         STANDARD ERROR FOR PCS2(V).
c
c      RTO(V)         RATIO OF PCS2(V)/PCS1(V).
c
c      NWIN1(L,M)     NUMBER OF WINS WITH METHOD I FOR POPULATION
c                     L WITH M VECTORS; L=2,3,...,K; M=1,2,...,100.
c
c      X(J,K)         OBSERVATION K FROM POPULATION J.
c
```

```
c      XCNT(J)          USED TO KEEP TRACK OF HOW MANY OBSERVATIONS
c                       FROM ALL OTHER POPULATIONS AN OBSERVATION
c                       FROM POPULATION J EXCEEDS.
c
c      XJ(J,M)          USED TO STORE HOW MANY OBSERVATIONS IN POP M
c                       ARE EXCEEDED BY CURRENT OBSERVATION FROM
c                       POP J.  PRODUCT OF XJ(J,M) FOR M=2,...,K;
c                       M NOT EQUAL TO K RESULTS IN TOTAL XCNT(J)
c                       FOR A SINGLE OBSERVATION FROM POP J.
c
c      FC(J)            SUM OF XCNT(J) OVER ALL OBSERVATIONS FOR
c                       POP J.
c
C****************************************************************
C****************************************************************
c
       PROGRAM MSP
c
c
       REAL LAMBDA,MU,X(5,100),XJ(5,5),XCNT(5),FC(5)
       REAL PCS1(100),PCS2(100),SE1(100),SE2(100),RTO(100)
       INTEGER*4 JX1(3),JX2(3),JX3(3),JX4(3),JX5(3)
       INTEGER*4 JX6(3),JX7(3),JX8(3),JX9(3),JX10(3)
       INTEGER NWIN1(5,100)
c
c      FILES TO OUTPUT RESULTS TO
c
       OPEN(4,FILE='exk5t20.dat',STATUS='UNKNOWN')
c
c      INITIALIZE VARIABLES
c
c      IDIST=1 IF EXPONENTIAL, 2 IF CONTINUOUS UNIFORM
c      3 IF GAMMA
c
       NREPS=100000
       NVECT=2
       NPOP=5
       LAMBDA=1.0
       MU=1.5885
       A=1.0
       B=1.1
       AX=3.0
```

```fortran
      BX=1.3061
      AY=3.0
      BY=1.0
      IDIST=1
c
c     SETUP CONSTANTS FOR GAMMA RV GENERATION
c
      AB=1.0/SQRT(2*AX-1)
      AR=1.0/SQRT(2*AY-1)
      BB=AX-LOG(4.0)
      BR=AY-LOG(4.0)
      QB=AX+1.0/AB
      QR=AY+1.0/AR
      TB=4.5
      TR=4.5
      DB=1.0+LOG(TB)
      DR=1.0+LOG(TR)
c
c     SEEDS FOR L'ECUYER'S RN GENERATOR
c
      JX1(2)=748932582
      JX1(3)=639287
      JX2(2)=64298628
      JX2(3)=196998
      JX3(2)=40689408
      JX3(3)=122595154
      JX4(2)=680620100
      JX4(3)=335083118
      JX5(2)=64918046
      JX5(3)=812715188
c
5     DO 10 I=1,NVECT
          PCS1(I)=0
          PCS2(I)=0
 10   CONTINUE
      DO 2000 I=I,NREPS
         DO 30 J=1,NPOP
            FC(J)=0
            DO 20 K=1,NVECT
               NWIN1(J,K)=0
 20   CONTINUE
 30   CONTINUE
```

```
c
c     RANDOM VARIATE GENERATION DONE HERE.   SEPARATE SET OF RN
c     STREAMS FOR EACH POPULTION.
c
      IF(IDIST.EQ.2)GO TO 65
      IF(IDIST.EQ.3)GO TO 81
c
c  EXPONENTIAL RANDOM VARIATES
c
         DO 60 K=1,NVECT
            X(1,K)=-(1/LAMBDA)*LOG(1-RUNF(JX1))
            X(2,K)=-(1/MU)*LOG(1-RUNF(JX2))
            IF(NPOP.EQ.2)GO TO 60
            X(3,K)=-(1/MU)*LOG(1-RUNF(JX3))
            IF(NPOP.EQ.3)GO TO 60
            X(4,K)=-(1/MU)*LOG(1-RUNF(JX4))
            IF(NPOP.EQ.4)GO TO 60
            X(5,K)=-(1/MU)*LOG(1-RUNF(JX5))
  60     CONTINUE
         GO TO 89
c
c  CONTINOUS UNIFORM RANDOM VARIATES
c
  65     DO 80 K=1,NVECT
            X(1,K)=B*RUNF(JX1)
            X(2,K)=A*RUNF(JX2)
            IF(NPOP.EQ.2)GO TO 80
            X(3,K)=A*RUNF(JX3)
            IF(NPOP.EQ.3)GO TO 80
            X(4,K)=A*RUNF(JX4)
            IF(NPOP.EQ.4)GO TO 80
            X(5,K)=A*RUNF(JX5)
  80     CONTINUE
      GO TO 89
c
c  GAMMA RANDOM VARIATES
c
  81  DO 88 K=1,NVECT
  82     U1=RUNF(JX1)
         U2=RUNF(JX2)
         V=AB*LOG(U1/(1-U1))
         Y=AX*EXP(V)
```

120

```
              Z=(U1**2)*U2
              W=BB+QB*V-Y
              IF(W+DB-TB*Z.GE.0)THEN
                  XT=Y
              ELSE IF(W.GE.LOG(Z))THEN
                  XT=Y
              ELSE
                  GO TO 82
              END IF
              X(1,K)=BX*XT
      83      U1=RUNF(JX3)
              U2=RUNF(JX4)
              V=AR*LOG(U1/(1-U1))
              Y=AY*EXP(V)
              Z=(U1**2)*U2
              W=BR+QR*V-Y
              IF(W+DR-TR*Z.GE.0)THEN
                  OT=Y
              ELSE IF(W.GE.LOG(Z))THEN
                  OT=Y
              ELSE
                  GO TO 83
              END IF
              X(2,K)=BY*OT
              IF(NPOP.EQ.2)GO TO 88
      84      U1=RUNF(JX5)
              U2=RUNF(JX6)
              V=AR*LOG(U1/(1-U1))
              Y=AY*EXP(V)
              Z=(U1**2)*U2
              W=BR+QR*V-Y
              IF(W+DR-TR*Z.GE.0)THEN
                  OT=Y
              ELSE IF(W.GE.LOG(Z))THEN
                  OT=Y
              ELSE
                  GO TO 84
              END IF
              X(3,K)=BY*OT
              IF(NPOP.EQ.3)GO TO 88
      85      U1=RUNF(JX7)
              U2=RUNF(JX8)
```

```
                V=AR*LOG(U1/(1-U1))
                Y=AY*EXP(V)
                Z=(U1**2)*U2
                W=BR+QR*V-Y
                IF(W+DR-TR*Z.GE.0)THEN
                    OT=Y
                ELSE IF(W.GE.LOG(Z))THEN
                    OT=Y
                ELSE
                    GO TO 85
                END IF
                X(4,K)=BY*OT
                IF(NPOP.EQ.4)GO TO 88
86              U1=RUNF(JX9)
                U2=RUNF(JX10)
                V=AR*LOG(U1/(1-U1))
                Y=AY*EXP(V)
                Z=(U1**2)*U2
                W=BR+QR*V-Y
                IF(W+DR-TR*Z.GE.0)THEN
                    OT=Y
                ELSE IF(W.GE.LOG(Z))THEN
                    OT=Y
                ELSE
                    GO TO 86
                END IF
                X(5,K)=BY*OT
 88     CONTINUE
c
c                       METHOD I (BEM)
c       SIMPLY NEED TO COUNT NUMBER OF TIMES EACH Xi IS BEST IN
c       SAME VECTOR.
c
 89         DO 100 K=1,NVECT
                BEST=X(1,K)
                DO 90 J=2,NPOP
                    TM=AMAX1(BEST,X(J,K))
                    IF(TM.GT.BEST)BEST=TM
 90             CONTINUE
                DO 95 L=1,NPOP
                    IF(X(L,K).EQ.TM)NWIN1(L,K)=NWIN1(L,K)+1
 95             CONTINUE
```

```
100     CONTINUE
c
c    IF BEST POPULATION (#1) WINS OUTRIGHT, INCREASE PCS BY ONE.
c    IF ANOTHER POPULATION WINS, DO NOT INCREASE PCS.
c    FOR TIES INCLUDING BEST POPULATION, NEED TO KEEP TRACK OF THE
c    NUMBER OF TIES AND INCREASE PCS BY 1/NTIES.
c
      DO 105 K=2,NVECT
         DO 102 J=1,NPOP
         NWIN1(J,K)=NWIN1(J,K)+NWIN1(J,K-1)
102   CONTINUE
105   CONTINUE
      DO 111 K=1,NVECT
      NTIES=1
      DO 110 J=2,NPOP
         IF(NWIN1(1,K).LT.NWIN1(J,K))THEN
            PCS1(K)=PCS1(K)
            GO TO 111
         ELSE
            IF(NWIN1(1,K).EQ.NWIN1(J,K))NTIES=NTIES+1
            IF(J.EQ.NPOP)PCS1(K)=PCS1(K)+1.0/(NTIES)
         END IF
110   CONTINUE
111   CONTINUE
c
c                    METHOD II (AVC)
c
      DO 1180 NM2=1,NVECT
      CALL SORTV(NPOP,NM2,X)
      DO 1150 K=1,NM2
         DO 1148 J=1,NPOP
            XCNT(J)=1
            DO 1146 M=1,NPOP
               DO 1142 L=1,NM2
                  IF(M.NE.J.AND.X(J,K).GT.X(M,L))THEN
                     XJ(J,M)=NM2-(L-1)
                     GO TO 1145
                  ELSE
                     IF(L.EQ.NM2)XJ(J,M)=0
                  END IF
1142           CONTINUE
1145           IF(M.NE.J)XCNT(J)=XCNT(J)*XJ(J,M)
```

123

```
1146        CONTINUE
1148     CONTINUE
      DO 1149 J=1,NPOP
          FC(J)=FC(J)+XCNT(J)
1149 CONTINUE
1150 CONTINUE
      NT=1
      DO 1170 J=2,NPOP
          IF(FC(1).LT.FC(J))THEN
              PCS2(NM2)=PCS2(NM2)
              GO TO 1180
          ELSE
              IF(FC(1).EQ.FC(J))NT=NT+1
              IF(J.EQ.NPOP)PCS2(NM2)=PCS2(NM2)+1.0/NT
          END IF
1170 CONTINUE
1180 CONTINUE
2000 CONTINUE
      DO 2010 I=1,NVECT
          PCS1(I)=PCS1(I)/REAL(NREPS)
          PCS2(I)=PCS2(I)/REAL(NREPS)
          SE1(I)=(PCS1(I)*(1.0-PCS1(I))/REAL(NREPS))**.5
          SE2(I)=(PCS2(I)*(1.0-PCS2(I))/REAL(NREPS))**.5
          RTO(I)=PCS2(I)/PCS1(I)
C
C  FILE 4 SETUP TO READ DIRECTLY INTO MINITAB OR OTHER PROGRAM
C
      WRITE(4,2050)NPOP,I,PCS1(I),PCS2(I),RTO(I),SE1(I),SE2(I)
C
 2010 CONTINUE
 2050 FORMAT(1X,2I4,2X,5(F8.6,2X))
      STOP
      END
C
      SUBROUTINE SORTV (K,N,Y)
C
      INTEGER K,N
      REAL Y(5,100)
C
      DO 30 L=1,K
      DO 20 J=2,N
        TEMP=Y(L,J)
```

```fortran
          I=J-1
10        IF(I.EQ.0)GO TO 20
          IF(TEMP.GT.Y(L,I))THEN
              Y(L,I+1)=Y(L,I)
              Y(L,I)=TEMP
              I=I-1
              GO TO 10
          END IF
20     CONTINUE
30     CONTINUE
       RETURN
       END
c
       FUNCTION RUNF(JX)
c**************************************************************
c    L'Ecuyer's generator as given as unifl in BF&S, p. 332
c
c    inputs:
c        jx(2), jx(3) = two random integers with:
c                       0 < jx(2) < 2147483563
c                       0 < jx(3) < 2147483399
c
c    outputs:
c        jx(1) pseudorandom integer, 0 < jx(3) < 2147483563
c              period of jx(1) is about 2.3e18
c        jx(2) = pseudorandom integer, 0 < jx(2) < 2147483563
c        jx(3) = pseudorandom integer, 0 < jx(2) < 2147483399
c        unifl = pseudorandom real, 0. < unifl < 1.
c
c**************************************************************
       INTEGER*4 JX, K
       DIMENSION JX(3)
c
c get next term in the first stream = 40014*jx(2) mod 2147483563
       K = JX(2)/53668
       JX(2) = 40014*(JX(2) - K*53668) - K*12211
       IF (JX(2) .LT. 0) JX(2) = JX(2) + 2147483563
c
c get next term in the second stream = 40692*jx(3) mod 2147483399
       K = JX(3)/52774
       JX(3) = 40692*(JX(3) - K*52774) - K*3791
       IF (JX(3) .LT. 0) JX(3) = JX(3) + 2147483399
```

```
c
c set jx(1) = ((jx(3) + 2147483562 - jx(2)) mod 2147483562) + 1
      K = JX(3) - JX(2)
      IF (K .LE. 0) K = K + 2147483562
c
c put the combination back into jx(1)
      JX(1) = K
c
c put it on the interval (0.,1.)
      RUNF = K*4.656613E-10
      RETURN
      END
C****************************************************************
C****************************************************************
C****************************************************************
c
c  THIS IS PROGRAM FOR MSP USING BERNOULLI POPULATIONS.
c  SETUP FOR MAX OF 5 POPULATIONS AND 50 VECTORS.
c  NOTE THIS CODE BREAKS ALL TIES WITHIN A VECTOR BY A RANDOM DRAW
c  RATHER THAN ASSIGNING A FRACTION OF A WIN.
c
c  BERNOULLI PARAMETERS:  PX = BERNOULLI SUCCESS PROBABILITY FOR
c  FIRST POPULATION (ASSUMED TO BE BEST), PY = BERNOULLI SUCCESS
c  PROBABILITY FOR ALL REMAINING POPULATIONS.
c
      PROGRAM MNBERN
c
c
      REAL PCS1(50),PCS2(50),SE1(50),SE2(50)
      PARAMETER(PX=.5,PY=.4264,NREPS=10000)
      INTEGER*4 JX1(3),JX2(3),JX3(3),JX4(3),JX5(3),LX(3),MX(3)
      REAL X(5,50),P(5),NWIN1(5,50),NWIN2(5,50)
c
c     FILES TO OUTPUT RESULTS TO
c
      OPEN(4,FILE='bk5t12.dat',STATUS='UNKNOWN')
c
      NPOP=5
      NVECT=50
      DO 10 I=1,50
         PCS1(I)=0
         PCS2(I)=0
```

```fortran
      10  CONTINUE
c
c     INITIALIZE VARIABLES
c
c     SEEDS FOR L'ECUYER'S RN GENERATOR
c
          JX1(2)=748932582
          JX1(3)=639287
          JX2(2)=64298628
          JX2(3)=196998
          JX3(2)=40689408
          JX3(3)=122595154
          JX4(2)=680620100
          JX4(3)=335083118
          JX5(2)=64918046
          JX5(3)=812715188
          LX(2)=427126672
          LX(3)=699944973
          MX(2)=257567734
          MX(3)=189120895
          DO 2000 I=1,NREPS
c
             DO 30 K=1,NVECT
                DO 20 J=1,NPOP
                   NWIN1(J,K)=0
                   NWIN2(J,K)=0
      20          CONTINUE
      30       CONTINUE
c
c     RANDOM VARIATE GENERATION DONE HERE.
c
      82     DO 87 K=1,50
               IF(RUNF(JX1).LE.PX)THEN
                  X(1,K)=1.0
               ELSE
                  X(1,K)=0.0
               END IF
               IF(RUNF(JX2).LE.PY)THEN
                  X(2,K)=1.0
               ELSE
                  X(2,K)=0.0
               END IF
```

```fortran
              IF(NPOP.EQ.2)GO TO 87
              IF(RUNF(JX3).LE.PY)THEN
                  X(3,K)=1.0
              ELSE
                  X(3,K)=0.0
              END IF
              IF(NPOP.EQ.3)GO TO 87
              IF(RUNF(JX4).LE.PY)THEN
                  X(4,K)=1.0
              ELSE
                  X(4,K)=0.0
              END IF
              IF(NPOP.EQ.4)GO TO 87
              IF(RUNF(JX5).LE.PY)THEN
                  X(5,K)=1.0
              ELSE
                  X(5,K)=0.0
              END IF
  87    CONTINUE
c
c                       METHOD I (BEM)
c       SIMPLY NEED TO COUNT NUMBER OF TIMES EACH Xi IS BEST IN
c       SAME VECTOR.  NOTE WITH USE OF BERNOULLI POPULATIONS HAVE
c       INTRODUCED ALOT OF TIES THAT MUST BE DEALT WITH.
c
c  THIS SECTION ADDED TO TAKE CARE OF TIES WITH BERNOULLI POPS.
c  SUM IS ACROSS EACH VECTOR.  BASIC LOGIC IN THIS SECTION IS IF
c  THERE ARE ANY TIES, DRAW A RANDOM NUMBER (PRN BELOW) AND ASSIGN
c  FIRST FRACTION (0 - 1/SUM) BASED ON NUMBER OF TIES TO FIRST POP
c  TIED AS WINNING (COULD BE ALL POPULATIONS TIED AT ZERO) AND
c  NEXT FRACTION (1/SUM-2/SUM) TO NEXT WINNING POPULATION, ETC.
c
c  IO - FLAG SET TO 1 IF SUM IS ZERO, LEFT AT ZERO OTHERWISE
c  N1 - COUNTER FOR NUMBER OF POPULATIONS WITH A '1' IN A VECTOR
c  PRN - RANDOM NUMBER USED TO PICK SINGLE BEST POP FOR VECTOR
c
  101     DO 106 K=1,NVECT
              SUM=0
              IO=0
              N1=0
              DO 103 L=1,NPOP
                  SUM=SUM+X(L,K)
```

```
103         CONTINUE
         IF(SUM.EQ.0)THEN
            SUM=REAL(NPOP)
            IO=1
         END IF
         PRN=RUNF(LX)
         IF(SUM.EQ.1)PRN=0
         DO 104 L=1,NPOP
            IF(X(L,K).EQ.0.AND.IO.EQ.0)GO TO 104
            N1=N1+1
            IF(PRN.LT.(N1/SUM))THEN
               NWIN1(L,K)=NWIN1(L,K)+1
               GO TO 106
            END IF
104      CONTINUE
106      CONTINUE
      DO 108 K=2,NVECT
         DO 107 J=1,NPOP
            NWIN1(J,K)=NWIN1(J,K)+NWIN1(J,K-1)
107      CONTINUE
108   CONTINUE
c
c     IF BEST POPULATION (#1) WINS OUTRIGHT, INCREASE PCS BY ONE.
c     IF ANOTHER POPULATION WINS, DO NOT INCREASE PCS.
c     FOR TIES INCLUDING BEST POPULATION, NEED TO KEEP TRACK OF THE
c     NUMBER OF TIES AND INCREASE PCS BY 1/NTIES.
c
      DO 111 K=1,NVECT
         NTIES=1
         DO 110 J=2,NPOP
            IF(NWIN1(1,K).LT.NWIN1(J,K))GO TO 111
            IF(NWIN1(1,K).EQ.NWIN1(J,K))NTIES=NTIES+1
            IF(J.EQ.NPOP)PCS1(K)=PCS1(K)+1.0/REAL(NTIES)
110      CONTINUE
111   CONTINUE
c
c                     METHOD II (AVC)
c     COMPARE EACH Xi WITH EVERY POSSIBLE VECTOR V**K
c     COMPARISONS. THE LOOP INDEXES GET RATHER CONFUSING
c     HERE, BUT ACTUALLY FORM ALL V**K POSSIBLE PSEDUO-VECTORS
c     FOR AVC. TO PARALLEL BEM IN TERMS OF BREAKING TIES NEED
c     TO CONSTRUCT EACH PSEUDO-VECTOR.
```

```
c
      GO TO(200,300,400,500)(NPOP-1)
c
c     LOOPS FOR NPOP=2
c
 200  DO 291 NS=1,NVECT
      DO 290 L2=1,NS
         DO 220 L1=1,NS
               IF(L2.LT.NS.AND.L1.LT.NS)GO TO 220
               SUM=0
               IO=0
               N1=0
               P(1)=X(1,L2)
               P(2)=X(2,L1)
               SUM=P(1)+P(2)
               IF(SUM.EQ.0)THEN
                   SUM=REAL(NPOP)
                   IO=1
               END IF
               PRN=RUNF(MX)
               IF(SUM.EQ.1)PRN=0
               DO 210 L=1,NPOP
                   IF(P(L).EQ.0.AND.IO.EQ.0)GO TO 210
                   N1=N1+1
                   IF(PRN.LT.(N1/SUM))THEN
                       NWIN2(L,NS)=NWIN2(L,NS)+1
                       GO TO 220
                   END IF
 210              CONTINUE
 220        CONTINUE
 290  CONTINUE
 291  CONTINUE
          GO TO 1100
c
c     LOOPS FOR NPOP=3
c
 300  DO 391 NS=1,NVECT
      DO 390 L3=1,NS
      DO 330 L2=1,NS
         DO 320 L1=1,NS
               IF(L3.LT.NS.AND.L2.LT.NS.AND.L1.LT.NS)GO TO 320
               SUM=0
```

130

```
                    IO=0
                    N1=0
                    P(1)=X(1,L3)
                    P(2)=X(2,L2)
                    P(3)=X(3,L1)
                    SUM=P(1)+P(2)+P(3)
                    IF(SUM.EQ.0)THEN
                        SUM=REAL(NPOP)
                        IO=1
                    END IF
                    PRN=RUNF(MX)
                    IF(SUM.EQ.1)PRN=0
                    DO 310 L=1,NPOP
                        IF(P(L).EQ.0.AND.IO.EQ.0)GO TO 310
                        N1=N1+1
                        IF(PRN.LT.(N1/SUM))THEN
                            NWIN2(L,NS)=NWIN2(L,NS)+1
                            GO TO 320
                        END IF
 310                CONTINUE
 320        CONTINUE
 330  CONTINUE
 390  CONTINUE
 391  CONTINUE
          GO TO 1100
c
c    LOOPS FOR NPOP=4
c
 400  DO 491 NS=1,NVECT
      DO 490 L4=1,NS
      DO 440 L3=1,NS
      DO 430 L2=1,NS
          DO 420 L1=1,NS
              IF(L4.LT.NS.AND.L3.LT.NS.AND.
     +          L2.LT.NS.AND.L1.LT.NS)GO TO 420
                SUM=0
                IO=0
                N1=0
                P(1)=X(1,L4)
                P(2)=X(2,L3)
                P(3)=X(3,L2)
                P(4)=X(4,L1)
```

```fortran
                    SUM=P(1)+P(2)+P(3)+P(4)
                    IF(SUM.EQ.0)THEN
                       SUM=REAL(NPOP)
                       I0=1
                    END IF
                    PRN=RUNF(MX)
                    IF(SUM.EQ.1)PRN=0
                    DO 410 L=1,NPOP
                       IF(P(L).EQ.0.AND.I0.EQ.0)GO TO 410
                       N1=N1+1
                       IF(PRN.LT.(N1/SUM))THEN
                          NWIN2(L,NS)=NWIN2(L,NS)+1
                          GO TO 420
                       END IF
 410                CONTINUE
 420                CONTINUE
 430    CONTINUE
 440    CONTINUE
 490    CONTINUE
 491    CONTINUE
           GO TO 1100
c
c    LOOPS FOR NPOP=5
c
 500    DO 591 NS=1,NVECT
        DO 590 L5=1,NS
        DO 580 L4=1,NS
        DO 570 L3=1,NS
        DO 530 L2=1,NS
           DO 520 L1=1,NS
              IF(L5.LT.NS.AND.L4.LT.NS.AND.L3.LT.NS.
     +           AND.L2.LT.NS.AND.L1.LT.NS)GO TO 520
                 SUM=0
                 I0=0
                 N1=0
                 P(1)=X(1,L5)
                 P(2)=X(2,L4)
                 P(3)=X(3,L3)
                 P(4)=X(4,L2)
                 P(5)=X(5,L1)
                 SUM=P(1)+P(2)+P(3)+P(4)+P(5)
                 IF(SUM.EQ.0)THEN
```

132

```fortran
                    SUM=REAL(NPOP)
                    IO=1
                END IF
                PRN=RUNF(MX)
                IF(SUM.EQ.1)PRN=0
                DO 510 L=1,NPOP
                    IF(P(L).EQ.O.AND.IO.EQ.0)GO TO 510
                    N1=N1+1
                    IF(PRN.LT.(N1/SUM))THEN
                        NWIN2(L,NS)=NWIN2(L,NS)+1
                        GO TO 520
                    END IF
510             CONTINUE
520             CONTINUE
530   CONTINUE
570   CONTINUE
580   CONTINUE
590   CONTINUE
591   CONTINUE
c
1100  CONTINUE
      DO 1105 K=2,NVECT
          DO 1102 J=1,NPOP
              NWIN2(J,K)=NWIN2(J,K)+NWIN2(J,K-1)
1102      CONTINUE
1105  CONTINUE
      DO 1111 K=1,NVECT
          NTIES=1
          DO 1110 J=2,NPOP
              IF(NWIN2(1,K).LT.NWIN2(J,K))GO TO 1111
              IF(NWIN2(1,K).EQ.NWIN2(J,K))NTIES=NTIES+1
              IF(J.EQ.NPOP)PCS2(K)=PCS2(K)+1.0/NTIES
1110      CONTINUE
1111  CONTINUE
2000  CONTINUE
      DO 2015 I=1,NVECT
          PCS2(I)=PCS2(I)/REAL(NREPS)
          PCS1(I)=PCS1(I)/REAL(NREPS)
          SE1(I)=(PCS1(I)*(1.0-PCS1(I))/REAL(NREPS))**.5
          SE2(I)=(PCS2(I)*(1.0-PCS2(I))/REAL(NREPS))**.5
          RTO=PCS2(I)/PCS1(I)
c
```

```
c  FILE 4 SETUP TO READ DIRECTLY INTO MINITAB OR OTHER SPREADSHEET
c
      WRITE(4,2050)NPOP,I,PCS1(I),PCS2(I),RTO,SE1(I),SE2(I)
c
2015  CONTINUE
2050  FORMAT(1X,2I4,2X,5(F8.6,2X))
      STOP
      END
c
C********************************************************************
C********************************************************************
C********************************************************************
c
c  PROGRAM TO GENERATE STANDARD MLE ESTIMATES FOR MULTINOMIAL
c  SUCCESS PROBABILITIES AND NEW AVC ESTIMATES.  GATHERING
c  DATA TO COMPARE THE VARIANCES BETWEEN THE TWO ESTIMATES
c  AND TO CONSTRUCT CONFIDENCE INTERVALS.
c  SETUP FOR K=2,3, AND 5 POPULATIONS UP TO V=200.
c  INITIALLY SET UP TO ESTIMATE ONLY P1 (BEST POP),
c  ALTHOUGH VARIABLES ARE DIMENSIONED TO CALCULATE ALL Pj'S.
c  ONLY MODELS EXPONENTIAL POPULATIONS.
c
C********************************************************************
C********************************************************************
c                     VARIABLE TABLE
c
c  NOTE:  ONLY VARIABLES UNIQUE TO THIS PROGRAM DEFINED HERE.
c         SEE VARIABLE TABLE IN PROGRAM MSP FOR MORE.
c
c     NAME           DESCRIPTION
c
c     AP(J)          AVC ESTIMATE FOR Pj (P-BARj).
c
c     BP(J)          BEM (OR MLE) ESTIMATE FOR Pj (P-HATj).
c
c     VK             V**K.
c
c     PA(J)          ACTUAL Pj FOR POPULATION J.
c
c     NL1(I)         COUNT USED IN CALCULATING FIRST TYPE OF
c                    U-STAT COVARIANCE TERM - L(1,0). INDEXED
c                    BY V.
```

```
c
c       NL1T            SUM OF NL1(I), I=1,...,V.
c
c       NL2(I)          COUNT USED IN CALCULATING SECOND TYPE OF
c                       U-STAT COVARIANCE TERM - L(0,1).
c
c       NL2T            SUM OF NL2(I), I=1,...,V.
c
c       NV              TOTAL NUMBER OF PSEUDO-VECTOR PAIRS WITH A
c                       SINGLE COMMON ELEMENT FOR GIVEN K AND V.
c                       USED IN U-STAT COVARIANCE CALCUALTIONS.
c
c       VAP(J)          VARIANCE OF P-BARj.
c
c       VBP(J)          VARIANCE OF P-HATj.
c
c       AHW             HALF-WIDTH OF AVC CONFIDENCE INTERVAL.
c
c       BHW             HALF-WIDTH OF MLE CONFIDENCE INTERVAL.
c
c       ALL(J)          AVC LOWER-CONFIDENCE LIMIT FOR P-BARj.
c
c       AUL(J)          AVC UPPER-CONFIDENCE LIMIT FOR P-BARj.
c
c       BLL(J)          MLE LOWER-CONFIDENCE LIMIT FOR P-HATj.
c
c       BUL(J)          MLE UPPER-CONFIDENCE LIMIT FOR P-HATj.
c
c       AHIT(J)         NUMBER/PCT OF AVC C.I. THAT CAPTURE PA(J).
c
c       BHIT(J)         NUMBER/PCT OF MLE C.I. THAT CAPTURE PA(J).
c
c       WA(J)           AVERAGE AVC C.I. WIDTH FOR P-BARj.
c
c       WB(J)           AVERAGE MLE C.I. WIDTH FOR P-HATj.
c
C****************************************************************
c
        PROGRAM CIK5
c
        REAL LAMBDA,MU,X(5,500),XCNT(5),FC(5)
        REAL ALL(5),AUL(5),BLL(5),BUL(5),PA(5),AHIT(5),BHIT(5)
```

```fortran
      REAL XJ(5,5),NL1(200),NL1T,NL2(200),NL2T,NV,VK,NCP,NCJ
      REAL BHW(5),AHW(5),WB(5),WA(5),M3(200,200),M4(200,200)
      REAL VAP(5),VBP(5),AP(5),BP(5),TAP(5),M5(200,200)
      REAL NM3,NM5,M3MIN,M4MIN,M5MIN,M3MAX,M4MAX,M5MAX
      INTEGER*4 JX1(3),JX2(3),JX3(3),JX4(3),JX5(3)
      INTEGER NWIN1(5)
C
C     FILES TO OUTPUT RESULTS TO
C

      OPEN(4,FILE='cik5v20a.dat',STATUS='UNKNOWN')
C
C     INITIALIZE VARIABLES
C
      NREPS=10000
      NVECT=20
      NPOP=5
      LAMBDA=1.0
      MU=1.1227
C
C  ADDITIONAL VARIABLES TO INITIALIZE FOR CONFIDENCE INTERVAL RUNS
C
      Z95=1.96
       PA(1)=.2308
       PA(2)=.1923
       PA(3)=.1923
       PA(4)=.1923
       PA(5)=.1923
C
C     SEEDS FOR L'ECUYER'S RN GENERATOR
C
      JX1(2)=748932582
      JX1(3)=639287
      JX2(2)=64298628
      JX2(3)=196998
      JX3(2)=40689408
      JX3(3)=122595154
      JX4(2)=680620100
      JX4(3)=335083118
      JX5(2)=64918046
      JX5(3)=812715188
C
```

```fortran
      CVL1=0
      CVL2=0
      CBP=0
      CAP=0
      CVBP=0
      CVAP=0
      BSSQ=0
      ASSQ=0
      DO 7 J=1,NPOP
         BHIT(J)=0
         AHIT(J)=0
         WB(J)=0
         WA(J)=0
7     CONTINUE
10    CONTINUE
      DO 2000 I=1,NREPS
         DO 30 J=1,NPOP
            BP(J)=0
            AP(J)=0
            FC(J)=0
            NWIN1(J)=0
30       CONTINUE
         DO 40 K=1,NVECT
            NL1(K)=1
            NL2(K)=1
40       CONTINUE
c
c     RANDOM VARIATE GENERATION DONE HERE.  SEPARATE SET OF RN
c     STREAMS FOR EACH POPULTION.
c
         DO 60 K=1,NVECT
            X(1,K)=-(1/LAMBDA)*LOG(1-RUNF(JX1))
            X(2,K)=-(1/MU)*LOG(1-RUNF(JX2))
            IF(NPOP.EQ.2)GO TO 60
            X(3,K)=-(1/MU)*LOG(1-RUNF(JX3))
            IF(NPOP.EQ.3)GO TO 60
            X(4,K)=-(1/MU)*LOG(1-RUNF(JX4))
            IF(NPOP.EQ.4)GO TO 60
            X(5,K)=-(1/MU)*LOG(1-RUNF(JX5))
60       CONTINUE
c
c                        METHOD I (BEM OR MLE)
```

```
c        FOR POINT ESTIMATES AND CONFIDENCE INTERVALS
c        DO NOT CALCULATE PCS.
c
 89        DO 100 K=1,NVECT
              BEST=X(1,K)
              DO 90 J=2,NPOP
                 TM=AMAX1(BEST,X(J,K))
                 IF(TM.GT.BEST)BEST=TM
 90           CONTINUE
              DO 95 L=1,NPOP
                 IF(X(L,K).EQ.TM)NWIN1(L)=NWIN1(L)+1
 95           CONTINUE
100        CONTINUE
           DO 115 J=1,NPOP
              BP(J)=NWIN1(J)/REAL(NVECT)
115        CONTINUE
           BPSQ=BP(1)*BP(1)
           BSSQ=BSSQ+BPSQ
c
c                          METHOD II  (AVC)
c
           NM2=NVECT
           NL1T=0
           NL2T=0
           DO 900 K1=1,NM2
              DO 800 K2=1,NM2
                 M3(K1,K2)=0
                 M4(K1,K2)=0
                 M5(K1,K2)=0
800           CONTINUE
900        CONTINUE
           IF(NM2.EQ.1)THEN
              NV=1
           ELSE
              RNM2=REAL(NM2)
              RNPOP=REAL(NPOP)
              NV=RNM2**RNPOP*(RNM2-1.0)**(RNPOP-1)/2.0
           END IF
           VK=RNM2**RNPOP
           CALL SORTV(NPOP,NM2,X)
        DO 1160 K=1,NM2
           DO 1148 J=1,NPOP
```

```fortran
                XCNT(J)=1
                DO 1146 M=1,NPOP
                   DO 1142 L=1,NM2
                      IF(M.NE.J.AND.X(J,K).GT.X(M,L))THEN
                         XJ(J,M)=NM2-(L-1)
                         GO TO 1145
                      ELSE
                         IF(L.EQ.NM2)XJ(J,M)=0
                      END IF
1142               CONTINUE
1145               IF(M.NE.J)XCNT(J)=XCNT(J)*XJ(J,M)
1146            CONTINUE
1148         CONTINUE
         DO 1149 J=1,NPOP
            FC(J)=FC(J)+XCNT(J)
1149 CONTINUE
c
c  THE COVARIANCE CALCULATIONS ARE DONE HERE.  QUITE DIFFERENT
c  FOR DIFFERENT K, SO HAVE SEPARATE SECTIONS BASED ON K. BASIC
c  IDEA IS TO COUNT UP NUMBER OF VECTOR PAIRS THAT MEET THE
c  CONDITIONS FOR THE PROBABILITY STATMENT ASSOCIATED WITH
c  L(1,0) OR L(0,1).  DO NOT HAVE TO FORM ALL V**K PSUEDO-REPS
c  TO SOLVE.  CAN COUNT UP NUMBER OF PAIRS WHERE CONDITIONS ARE
c  MET AND THEN SUBTRACT OFF ANY THAT HAVE COMMON ELEMENTS OTHER
c  THAN THE ONE DESIRED.
c
         IF(NPOP.EQ.2)THEN
            NL1(1)=XJ(1,2)*(XJ(1,2)-1)/2
            NL1(2)=(NM2-XJ(1,2))*(NM2-XJ(1,2)-1)/2
            IF(XJ(1,2).LE.1)NL1(1)=0
            IF(XJ(1,2).EQ.NM2)NL1(2)=0
            NL1T=NL1T+NL1(1)
            NL1T=NL1T+NL1(2)
            NL2(1)=(NM2-XJ(2,1))*(NM2-XJ(2,1)-1)/2
            NL2(2)=XJ(2,1)*(XJ(2,1)-1)/2
            IF(XJ(2,1).EQ.NM2)NL2(1)=0
            IF(XJ(2,1).LE.1)NL2(2)=0
            NL2T=NL2T+NL2(1)
            NL2T=NL2T+NL2(2)
            GO TO 1160
         END IF
         IF(NPOP.EQ.3)THEN
```

```
                NCJ=MAX(XJ(1,2),XJ(1,3))
                NCP=MIN(XJ(1,2),XJ(1,3))
                NL1(1)=(NCP*(NCP-1)/2)*(NCJ**2-NCJ)
                IF(XJ(1,2).LE.1.OR.XJ(1,3).LE.1)NL1(1)=0
                NL1T=NL1T+NL1(1)
            END IF
            DO 1158 K2=1,NM2
                IF(X(1,K).GT.X(2,K2))M3(K,K2)=XJ(1,3)
                IF(NPOP.EQ.3)GO TO 1158
                IF(X(1,K).GT.X(2,K2))M4(K,K2)=XJ(1,4)
                IF(X(1,K).GT.X(2,K2))M5(K,K2)=XJ(1,5)
 1158 CONTINUE
            IF(NPOP.EQ.5)THEN
                BU=XJ(1,2)
                BW=XJ(1,3)
                BY=XJ(1,4)
                BZ=XJ(1,5)
                WYMIN=MIN(BW,BY)
                WZMIN=MIN(BW,BZ)
                YZMIN=MIN(BY,BZ)
                NL1(1)=BU*(BU-1)/2*((BW*BY*BZ)**2-(BW*BY+BW*BZ+BY*BZ
       +        -WYMIN-WZMIN-YZMIN)*BW*BY*BZ)
                NL1T=NL1T+NL1(1)
            END IF
 1160 CONTINUE
            IF(NPOP.EQ.2)GO TO 1168
            NM3=0
            NM5=0
            DO 1165 K1=1,NM2
                DO 1163 K3=1,NM2
                    IF(K3.LE.K1)GO TO 1163
                    DO 1162 K2=1,NM2
                        IF(NPOP.EQ.5)GO TO 1161
                        NM3=M3(K1,K2)*M3(K3,K2)-MIN(M3(K1,K2),M3(K3,K2))
                        NL2T=NL2T+NM3
                        GO TO 1162
 1161                   M51=M3(K1,K2)*M4(K1,K2)*M5(K1,K2)
                        M52=M3(K3,K2)*M4(K3,K2)*M5(K3,K2)
                        M3MIN=MIN(M3(K1,K2),M3(K3,K2))
                        M4MIN=MIN(M4(K1,K2),M4(K3,K2))
                        M5MIN=MIN(M5(K1,K2),M5(K3,K2))
                        M3MAX=MAX(M3(K1,K2),M3(K3,K2))
```

```
                    M4MAX=MAX(M4(K1,K2),M4(K3,K2))
                    M5MAX=MAX(M5(K1,K2),M5(K3,K2))
                    A1=M3MAX*M4MAX-MIN(M3MAX,M4MAX)+1
                    A2=M3MAX*M5MAX-MIN(M3MAX,M5MAX)
                    A3=M4MAX*M5MAX-MIN(M4MAX,M5MAX)
                    NM5=M51*M52-(A1+A2+A3)*M3MIN*M4MIN*M5MIN
                    NL2T=NL2T+NM5
 1162          CONTINUE
 1163       CONTINUE
 1165 CONTINUE
 1168 CONTINUE
      DO 1169 J=1,NPOP
          TAP(J)=FC(J)/VK
          AP(J)=AP(J)+TAP(J)
 1169 CONTINUE
      APSQ=AP(1)*AP(1)
      ASSQ=ASSQ+APSQ
      CVL1=NL1T/NV
      CVL2=NL2T/NV
c
c THE FOLLOWING SECTION DOES FINAL CALCULATIONS FOR Pj'S AND CI'S
c
      CVL1=CVL1-AP(1)**2
      CVL2=CVL2-AP(1)**2
      VAP(1)=(CVL1+(NPOP-1)*CVL2)/NVECT
      VBP(1)=(BP(1)*(1.0-BP(1)))/NVECT
c
c  THESE ARE JUST RUNNING TOTALS FOR POINT ESTIMATES AND
c  VARIANCES
c
       CBP=CBP+BP(1)
       CAP=CAP+AP(1)
       CVBP=CVBP+VBP(1)
       CVAP=CVAP+VAP(1)
c
c     HERE IS WHERE THE FINAL CONFIDENCE INTERVAL CONSTRUCTION IS
c     DONE. THE ONLY DATA WE ARE GATHERING HERE IS WHETHER OR NOT
c     EACH AVC AND BEM CI IS A HIT OR NOT, AND THE WIDTH OF THE
c     INTERVAL.
c
      DO 1200 J=1,1
          BHW(J)=Z95*SQRT(VBP(J))
```

```
              BLL(J)=BP(J)-BHW(J)
              BUL(J)=BP(J)+BHW(J)
              WB(J)=WB(J)+2*BHW(J)
              IF(BLL(J).LE.PA(J).AND.BUL(J).GE.PA(J))BHIT(J)=BHIT(J)+1
              AHW(J)=Z95*SQRT(VAP(J))
              ALL(J)=AP(J)-AHW(J)
              AUL(J)=AP(J)+AHW(J)
              WA(J)=WA(J)+2*AHW(J)
              IF(ALL(J).LE.PA(J).AND.AUL(J).GE.PA(J))AHIT(J)=AHIT(J)+1
 1200   CONTINUE
 2000 CONTINUE
C
C  HERE WE CALCULATE THE PERCENTAGE OF HITS WE HAD WITH EACH METHOD
C  OF CONFIDENCE INTERVALS AND ALSO THE AVERAGE CI WIDTH
C
        DO 2200 J=1,1
              BHIT(J)=BHIT(J)/REAL(NREPS)
              WB(J)=WB(J)/REAL(NREPS)
              AHIT(J)=AHIT(J)/REAL(NREPS)
              WA(J)=WA(J)/REAL(NREPS)
 2200 CONTINUE
C
C  AS A CHECK ON THE BIAS OF OUR AVC VARIANCE ESTIMATE, CALCULATE
C  THE SAMPLE VARIANCE ASSOCIATED WITH EACH METHOD HERE.
C
        SVARB=(BSSQ-(CBP*CBP/NREPS))/(NREPS-1)
        SVARA=(ASSQ-(CAP*CAP/NREPS))/(NREPS-1)
        CBP=CBP/NREPS
        CAP=CAP/NREPS
        CVBP=CVBP/NREPS
        CVAP=CVAP/NREPS
        VBIASB=CVMP-SVARB
        VBIASA=CVAP-SVARA
C
C  FILE 4 SETUP TO PRINT RESULTS IN READABLE FASHION
C
        WRITE(4,*)'CONFIDENCE INTERVAL RESULTS EXP POPULATIONS'
        WRITE(4,*)'K=5 AND P1=0.2308'
        WRITE(4,*)
        WRITE(4,*)'BEM RESULTS '
        WRITE(4,*)'   V      AVG VAR    SAMPLE VAR  VAR BIAS'
        WRITE(4,2500)NVECT,CVBP,SVARB,VBIASB
```

```
      WRITE(4,*)' '
      WRITE(4,*)'  V    PCT HITS   AVG CI WIDTH'
      WRITE(4,2600)NVECT,BHIT(1),WB(1)
      WRITE(4,*)' '
      WRITE(4,*)'AVC RESULTS'
      WRITE(4,*)'  V     AVG VAR    SAMPLE VAR  VAR BIAS'
      WRITE(4,2500)NVECT,CVAP,SVARA,VBIASA
      WRITE(4,*)' '
      WRITE(4,*)'  V    PCT HITS   AVG CI WIDTH'
      WRITE(4,2600)NVECT,AHIT(1),WA(1)
 2500 FORMAT(1X,I4,2X,3(F8.4,4X))
 2600 FORMAT(1X,I4,2X,2(F8.4,4X))
      STOP
      END
```

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